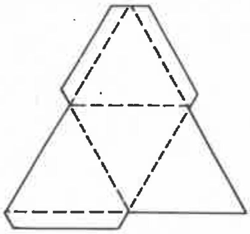
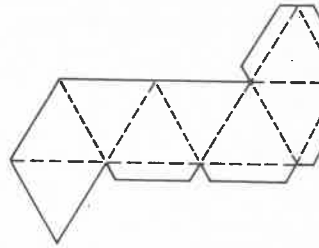
PATTERNS OF REGULAR POLYHEDRA

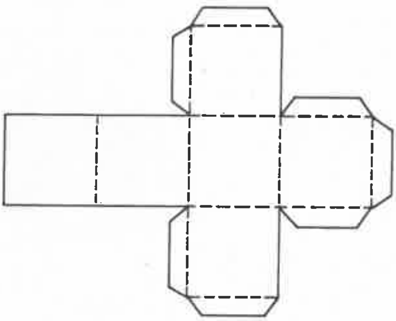
Tetrahedron



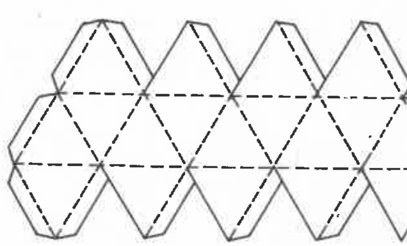
Octahedron



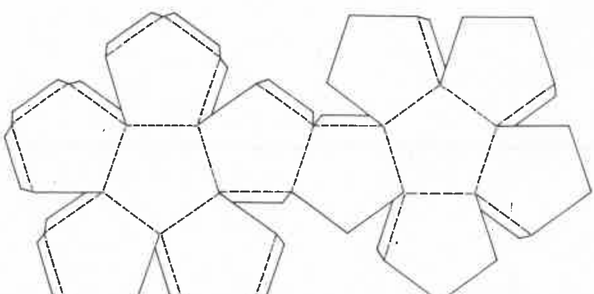
Hexahedron, or Cube



Icosahedron



Dodecahedron



To graph $\int_0^2 \int_{2x}^{x^2} (x - 2y)\, dy\, dx$

Use $\Rightarrow$ $y = 2x$ and $y = x^2$ for $x = 0$ to $x = 2$

if want to switch to $\int\int dx\, dy$ then $x = y/2$ and

+ The limits on y must be values of y that corre

to $x = 0$ and $x = 2$.

CALCULUS

Derivatives *

the following formulas u, v, w represent functions of x, while a, c, n represent fixed real bers. All arguments in the trigonometric functions are measured in radians, and all inverse nometric and hyperbolic functions represent principal values.

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(au) = a\frac{du}{dx}$$

$$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$ and so on to n factors

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}}\left(nv\frac{du}{dx} - mu\frac{dv}{dx}\right)$$

$$\frac{d}{dx}(u^n v^m) = u^{n-1}v^{m-1}\left(nv\frac{du}{dx} + mu\frac{dv}{dx}\right)$$

$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$$

Max + Min

(1) Find Z_x and Z_y + set them both equal to zero. Solve simultaneously to get Number values for x + y. See if there is a max or min of the function it occurs at this pt. of x + y.

(2) Find Z_{xx} and Z_{yy} + Z_{xy} ← This is the second derivitive of y taken) w/r to x.

(3) Plug into $Z_{xx} Z_{yy} - (Z_{xy})^2$

(a) and if >0 and Z_{xx} + Z_{yy} are neg = max

(b) if >0 but Z_{xx} + Z_{yy} are positive = min

(c) if <0 it is a "critical value which is neither a max or min

(d) if = 0 the test fails to give desired info

et $y = f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any ue x in their common domain. The differential for the function at such a value x is ordingly defined as

$$dy = d[f(x)] = \frac{dy}{dx}dx = \frac{d[f(x)]}{dx}dx = f'(x)dx$$

ch derivative formula has an associated differential formula. For example, formula 6 above ; the differential formula

) ntl osilate N=1 > N=-1

mi Sn D.M.E. SEQ

L.l = ±∞ ∴ Diverges

$$d(uvw) = uv\,dw + vw\,du + uw\,dv$$

DERIVATIVES (Continued)

15. $\frac{d^2}{dx^2}[f(u)] = \frac{df(u)}{du}\cdot\frac{d^2u}{dx^2} + \frac{d^2f(u)}{du^2}\cdot\left(\frac{du}{dx}\right)^2$

16. $\frac{d^n}{dx^n}[uv] = \binom{n}{0}v\frac{d^nu}{dx^n} + \binom{n}{1}\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}} + \binom{n}{2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}}$

$+ \cdots + \binom{n}{k}\frac{d^kv}{dx^k}\frac{d^{n-k}u}{dx^{n-k}} + \cdots + \binom{n}{n}u\frac{d^nv}{dx^n}$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ the binomial coefficient, n non-negative integer and $\binom{n}{0} = 1$.

17. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$ if $\frac{dx}{du} \neq 0$

18. $\frac{d}{dx}(\log_a u) = (\log_a e)\frac{1}{u}\frac{du}{dx}$

19. $\frac{d}{dx}(\log_e u) = \frac{1}{u}\frac{du}{dx}$

20. $\frac{d}{dx}(a^u) = a^u(\log_e a)\frac{du}{dx}$

21. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

22. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + (\log_e u)u^v\frac{dv}{dx}$

23. $\frac{d}{dx}(\sin u) = \frac{du}{dx}(\cos u)$

24. $\frac{d}{dx}(\cos u) = -\frac{du}{dx}(\sin u)$

25. $\frac{d}{dx}(\tan u) = \frac{du}{dx}(\sec^2 u)$

26. $\frac{d}{dx}(\cot u) = -\frac{du}{dx}(\csc^2 u)$

27. $\frac{d}{dx}(\sec u) = \frac{du}{dx}\sec u \cdot \tan u$

28. $\frac{d}{dx}(\csc u) = -\frac{du}{dx}\csc u \cdot \cot u$

29. $\frac{d}{dx}(\text{vers } u) = \frac{du}{dx}\sin u$

30. $\frac{d}{dx}(\text{arc sin } u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $\left(-\frac{\pi}{2} \le \text{arc sin } u \le \frac{\pi}{2}\right)$

(A) Partial Derivativ

① $du = u_x dx + u_y dy + u_z dz$

② if $V = \pi r^2 h$ (tuna can Prob.)

$dV = V_r dr + V_h dh = 2\pi rh\, dr + \pi$

③ Relative error:

$\frac{\text{Approximate value}}{\text{actual value}} = \frac{d}{V}$

(B) TOTAL DERIVATIVES FROM Parti

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}$

(C) Diff of Implicit Function

$\frac{dy}{dx} = -\left(\frac{\partial\phi/\partial x}{\partial\phi/\partial y}\right)$ & $\frac{\partial\phi}{\partial y} \neq 0$

DERIVATIVES (Continued)

$$\frac{d}{dx}(\text{arc cos } u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad (0 \le \text{arc cos } u \le \pi)$$

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1+u^2}\frac{du}{dx}, \qquad \left(-\frac{\pi}{2} < \text{arc tan } u < \frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc cot } u) = -\frac{1}{1+u^2}\frac{du}{dx}, \qquad (0 \le \text{arc cot } u \le \pi)$$

$$\frac{d}{dx}(\text{arc sec } u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \qquad \left(0 \le \text{arc sec } u < \frac{\pi}{2}, -\pi \le \text{arc sec } u < -\frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc csc } u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \qquad \left(0 < \text{arc csc } u \le \frac{\pi}{2}, -\pi < \text{arc csc } u \le -\frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc vers } u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}, \qquad (0 \le \text{arc vers } u \le \pi)$$

$$\frac{d}{dx}(\sinh u) = \frac{du}{dx}(\cosh u)$$

$$\frac{d}{dx}(\cosh u) = \frac{du}{dx}(\sinh u)$$

$$\frac{d}{dx}(\tanh u) = \frac{du}{dx}(\text{sech}^2\, u)$$

$$\frac{d}{dx}(\coth u) = -\frac{du}{dx}(\text{csch}^2\, u)$$

$$\frac{d}{dx}(\text{sech } u) = -\frac{du}{dx}(\text{sech } u \cdot \tanh u)$$

$$\frac{d}{dx}(\text{csch } u) = -\frac{du}{dx}(\text{csch } u \cdot \coth u)$$

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2+1})] = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2-1})] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad (u > 1, \cosh^{-1} u > 0)$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \qquad (u^2 < 1)$$

$$\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \qquad (u^2 > 1)$$

$$\frac{d}{dx}(\text{sech}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad (0 < u < 1, \text{sech}^{-1} u > 0)$$

$$\frac{d}{dx}(\text{csch}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

49. $\dfrac{d}{dq}\displaystyle\int_p^q f(x)\,dx = f(q),$ [p constant]

50. $\dfrac{d}{dp}\displaystyle\int_p^q f(x)\,dx = -f(p),$ [q constant]

51. $\dfrac{d}{da}\displaystyle\int_p^q f(x,a)\,dx = \int_p^q \frac{\partial}{\partial a}[f(x,a)]\,dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$

① PARTIAL FRACTIONS/RATIONAL FRACTIONS CASE

$\dfrac{X+C}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$

DEF NOT BASE (e)

$\lim_{x\to 0}(1+x)^{1/x}$ $\ln y = \lim_{x\to 0}$

$= \lim \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

$= \frac{1}{1+x} = \frac{1}{1} = 1$ $\ln y = 1$ $= e$

② Area Trapozoidal Rule:

$\dfrac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)\right]$

$\Delta x = \dfrac{x_n - x_0}{n} = \dfrac{\text{upper limit} - \text{lower limit}}{n\text{ (given)}}$

③ Rectangular Area

$= \int (f_1(x) - f_2(x))\,dx$

0·∞ $\lim_{x\to\pi/2} \sec 3x \cos 5x = \lim_{x\to\pi/2}$

$= \lim_{x\to\pi/2} \dfrac{+5\sin 5x}{+3\sin 3x} = -$

④ Polar Area:

$= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$

⑤ $Vol_{disk} = \pi \int r^2\,dh$

⑥ $Vol_{of\ shell} = 2\pi \int r \cdot h \cdot dr$

⑦ $s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $\sqrt{1+\frac{dx}{dy}}\,dy$ Arc length $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (}$

⑧ $S = 2\pi \int_{x_1}^{x_2} r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ Surface Area

INTEGRATION

: following is a brief discussion of some integration techniques. A more complete dis- n can be found in a number of good text books. However, the purpose of this introduction ply to discuss a few of the important techniques which may be used, in conjunction with tegral table which follows, to integrate particular functions.

matter how extensive the integral table, it is a fairly uncommon occurrence to find : table the exact integral desired. Usually some form of transformation will have to be . The simplest type of transformation, and yet the most general, is substitution. Simple ; of substitution, such as $y = ax$, are employed almost unconsciously by experienced of integral tables. Other substitutions may require more thought. In some sections of ibles, appropriate substitutions are suggested for integrals which are similar to, but not ly like, integrals in the table. Finding the right substitution is largely a matter of intuition xperience.

veral precautions must be observed when using substitutions:

: sure to make the substitution in the *dx* term, as well as everywhere else in the integral.
: sure that the function substituted is one-to-one and continuous. If this is not the case, e integral must be restricted in such a way as to make it true. See the example following.
'ith definite integrals, the limits should also be expressed in terms of the new dependent iriable. With indefinite integrals, it is necessary to perform the reverse substitution to otain the answer in terms of the original independent variable. This may also be done r definite integrals, but it is usually easier to change the limits.

cample:

$$\int \frac{x^4}{\sqrt{a^2 - x^2}}\, dx$$

ere we make the substitution $x = |a| \sin \theta$. Then $dx = |a| \cos \theta \, d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a|\sqrt{1 - \sin^2 \theta} = |a \cos \theta|$$

ice the absolute value signs. It is very important to keep in mind that a square root radical iys denotes the positive square root, and to assure the sign is always kept positive. Thus $^2 = |x|$. Failure to observe this is a common cause of errors in integration.

lotice also that the indicated substitution is not a one-to-one function, that is, it does not e a unique inverse. Thus we must restrict the range of θ in such a way as to make the ction one-to-one. Fortunately, this is easily done by solving for θ

$$\theta = \sin^{-1} \frac{x}{|a|}$$

l restricting the inverse sine to the principal values, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

'hus the integral becomes

$$\int \frac{a^4 \sin^4 \theta |a| \cos \theta \, d\theta}{|a| |\cos \theta|}$$

w, however, in the range of values chosen for θ, $\cos \theta$ is always positive. Thus we may remove absolute value signs from $\cos \theta$ in the denominator. (This is one of the reasons that the ncipal values of the inverse trigonometric functions are defined as they are.)

Then the $\cos\theta$ terms cancel, and the integral becomes

$$a^4 \int \sin^4\theta \, d\theta$$

By application of integral formulas 299 and 296, we integrate this to

$$-a^4 \frac{\sin^3\theta\cos\theta}{4} - \frac{3a^4}{8}\cos\theta\sin\theta + \frac{3a^4}{8}\theta + C$$

We now must perform the inverse substitution to get the result in terms of x. We have

$$\theta = \sin^{-1}\frac{x}{|a|}$$

$$\sin\theta = \frac{x}{|a|}$$

Then

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - \frac{x^2}{a^2}} = \pm\frac{\sqrt{a^2 - x^2}}{|a|}.$$

Because of the previously mentioned fact that $\cos\theta$ is positive, we may omit the $\pm$ sig[n]. The reverse substitution then produces the final answer

$$\int \frac{x^4}{\sqrt{a^2 - x^2}}\,dx = -\tfrac{1}{4}x^3\sqrt{a^2 - x^2} - \tfrac{3}{8}a^2x\sqrt{a^2 - x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{|a|} + C.$$

Any rational function of x may be integrated, if the denominator is factored into linea[r] and irreducible quadratic factors. The function may then be broken into partial fractions and the individual partial fractions integrated by use of the appropriate formula from th[e] integral table. See the section on partial fractions for further information.

Many integrals may be reduced to rational functions by proper substitutions. For example

$$z = \tan\frac{x}{2}$$

will reduce any rational function of the six trigonometric functions of x to a rational functio[n] of z. (Frequently there are other substitutions which are simpler to use, but this one will always work. See integral formula number 484.)

Any rational function of x and $\sqrt{ax + b}$ may be reduced to a rational function of z by making the substitution

$$z = \sqrt{ax + b}.$$

Other likely substitutions will be suggested by looking at the form of the integrand.

The other main method of transforming integrals is integration by parts. This involves applying formula number 5 or 6 in the accompanying integral table. The critical factor in this method is the choice of the functions u and v. In order for the method to be successful, $v = \int dv$ and $\int v\,du$ must be easier to integrate than the original integral. Again, this choice is largely a matter of intuition and experience.

Example:

$$\int x \sin x \, dx$$

Two obvious choices are $u = x$, $dv = \sin x\,dx$, or $u = \sin x$, $dv = x\,dx$. Since a preliminary mental calculation indicates that $\int v\,du$ in the second choice would be more, rather than less, nplicated than the original integral (it would contain x^2), we use the first choice.

$$u = x \qquad du = dx$$

$$dv = \sin x\, dx \qquad v = -\cos x$$

$$\int x \sin x\, dx = \int u\, dv = uv - \int v\, du = -x\cos x + \int \cos x\, dx$$

$$= \sin x - x \cos x$$

course, this result could have been obtained directly from the integral table, but it provides imple example of the method. In more complicated examples the choice of u and v may t be so obvious, and several different choices may have to be tried. Of course, there is no arantee that any of them will work.

Integration by parts may be applied more than once, or combined with substitution. A rly common case is illustrated by the following example.

Example:

$$\int e^x \sin x\, dx$$

Let

$$u = e^x \qquad \text{Then} \quad du = e^x\, dx$$

$$dv = \sin x\, dx \qquad v = -\cos x$$

$$\int e^x \sin x\, dx = \int u\, dv = uv - \int v\, du = -e^x \cos x + \int e^x \cos x\, dx$$

In this latter integral,

$$\text{let} \quad u = e^x \qquad \text{Then} \quad du = e^x\, dx$$

$$dv = \cos x\, dx. \qquad v = \sin x$$

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx = -e^x \cos x + \int u\, dv$$

$$= -e^x \cos x + uv - \int v\, du$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

'his looks as if a circular transformation has taken place, since we are back at the same ntegral we started from. However, the above equation can be solved algebraically for the equired integral:

$$\int e^x \sin x\, dx = \tfrac{1}{2}(e^x \sin x - e^x \cos x)$$

In the second integration by parts, if the parts had been chosen as $u = \cos x$, $dv = e^x\, dx$, we would indeed have made a circular transformation, and returned to the starting place. In general, when doing repeated integration by parts, one should never choose the function u at any stage to be the same as the function v at the previous stage, or a constant times the previous v.

The following rule is called the extended rule for integration by parts. It is the result of $n + 1$ successive applications of integration by parts.

If

$$g_1(x) = \int g(x)\,dx, \qquad g_2(x) = \int g_1(x)\,dx,$$

$$g_3(x) = \int g_2(x)\,dx, \ldots, g_m(x) = \int g_{m-1}(x)\,dx, \ldots,$$

then

$$\int f(x)\cdot g(x)\,dx = f(x)\cdot g_1(x) - f'(x)\cdot g_2(x) + f''(x)\cdot g_3(x) - + \cdots$$

$$+ (-1)^n f^{(n)}(x) g_{n+1}(x) + (-1)^{n+1} \int f^{(n+1)}(x) g_{n+1}(x)\,dx$$

A useful special case of the above rule is when $f(x)$ is a polynomial of degree n. The $f^{(n+1)}(x) = 0$, and

$$\int f(x)\cdot g(x)\,dx = f(x)\cdot g_1(x) - f'(x)\cdot g_2(x) + f''(x)\cdot g_3(x) - + \cdots + (-1)^n f^{(n)}(x) g_{n+1}(x) + ($$

Example:

If $f(x) = x^2$, $g(x) = \sin x$

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Another application of this formula occurs if

$$f''(x) = af(x) \quad \text{and} \quad g''(x) = bg(x),$$

where a and b are unequal constants. In this case, by a process similar to that used in th above example for $\int e^x \sin x\,dx$, we get the formula

$$\int f(x)g(x)\,dx = \frac{f(x)\cdot g'(x) - f'(x)\cdot g(x)}{b-a} + C$$

This formula could have been used in the example mentioned. Here is another example.

Example:

If $f(x) = e^{2x}$, $g(x) = \sin 3x$, then $a = 4, b = -9$, and

$$\int e^{2x} \sin 3x\,dx = \frac{3\,e^{2x}\cos 3x - 2\,e^{2x}\sin 3x}{-9-4} + C = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

The following additional points should be observed when using this table.

1. A constant of integration is to be supplied with the answers for indefinite integrals.
2. Logarithmic expressions are to base $e = 2.71828\ldots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.
3. All angles are measured in radians, and inverse trigonometric and hyperbolic functions represent principal values, unless otherwise indicated.
4. If the application of a formula produces either a zero denominator or the square root of a negative number in the result, there is usually available another form of the answer which avoids this difficulty. In many of the results, the excluded values are specified, but when such are omitted it is presumed that one can tell what these should be, especially when difficulties of the type herein mentioned are obtained.
5. When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes

ttle difficulty when the argument of the inverse trigonometric function is positive, since ıen all angles involved are in the first quadrant. However, if the argument is negative, pecial care must be used. Thus if $u > 0$,

$$\sin^{-1} u = \cos^{-1}\sqrt{1-u^2} = \csc^{-1}\frac{1}{u}, \text{ etc.}$$

Iowever, if $u < 0$,

$$\sin^{-1} u = -\cos^{-1}\sqrt{1-u^2} = -\pi - \csc^{-1}\frac{1}{u}, \text{ etc.}$$

iee the section on inverse trigonometric functions for a full treatment of the allowable ubstitutions.

n integrals 340–345 and some others, the right side includes expressions of the form

$$A \tan^{-1}[B + C \tan f(x)].$$

In these formulas, the $\tan^{-1}$ does not necessarily represent the principal value. Instead of ılways employing the principal branch of the inverse tangent function, one must instead use that branch of the inverse tangent function upon which $f(x)$ lies for any particular choice of x.

Example:

$$\int_0^{4\pi} \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\tan^{-1}\left.\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right]_0^{4\pi}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2\tan 2\pi+1}{\sqrt{3}} - \tan^{-1}\frac{2\tan 0+1}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{13\pi}{6} - \frac{\pi}{6}\right] = \frac{4\pi}{\sqrt{3}} = \frac{4\sqrt{3}\pi}{3}$$

ere

$$\tan^{-1}\frac{2\tan 2\pi+1}{\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{13\pi}{6},$$

nce $f(x) = 2\pi$; and

$$\tan^{-1}\frac{2\tan 0+1}{\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6},$$

nce $f(x) = 0$.

B_n and E_n where used in Integrals represents the Bernoulli and Euler numbers as defined in the tables contained from pages 408 to 414.

Laplace
15 translation theorem $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a} = F(s-a)$

$\mathcal{L}\{e^{5t}t^3\} = \mathcal{L}\{t^3\}_{s\to s-5} = \frac{3!}{s^4}\Big|_{s\to s-5} = \frac{6}{(s-5)^4}$

$\mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{\cos 4t\}_{s\to s+2} = \frac{s}{s^2+16}\Big|_{s\to s+2}$

INTEGRALS

ELEMENTARY FORMS

1. $\int a\,dx = ax$

2. $\int a \cdot f(x)\,dx = a \int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \frac{\phi(y)}{y'}\,dy, \qquad \text{where } y' = \frac{dy}{dx}$

4. $\int (u + v)\,dx = \int u\,dx + \int v\,dx, \qquad$ where u and v are any functions of x

5. $\int u\,dv = u \int dv - \int v\,du = uv - \int v\,du$

6. $\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

7. $\int x^n\,dx = \frac{x^{n+1}}{n+1}, \qquad \text{except } n = -1$

8. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x), \qquad (df(x) = f'(x)\,dx)$

9. $\int \frac{dx}{x} = \log x$

10. $\int \frac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)}, \qquad (df(x) = f'(x)\,dx)$

11. $\int e^x\,dx = e^x$

12. $\int e^{ax}\,dx = e^{ax}/a$

13. $\int b^{ax}\,dx = \frac{b^{ax}}{a \log b}, \qquad (b > 0)$

14. $\int \log x\,dx = x \log x - x$

15. $\int a^x \log a\,dx = a^x, \qquad (a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\int \ln|x|\,dx = x \ln$

$\int x^n \ln|x|\,dx = \frac{x^{n+1}}{n+1}\left(\ln|x| - \frac{1}{n+1}\right)$

INTEGRALS (Continued)

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{a+x}{a-x}, \quad (a^2 > x^2) \end{cases}$$

$$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a}\coth^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{x-a}{x+a}, \quad (x^2 > a^2) \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1}\frac{x}{|a|} \\ \text{or} \\ -\cos^{-1}\frac{x}{|a|}, \quad (a^2 > x^2) \end{cases}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

FORMS CONTAINING $(a + bx)$

For forms containing $a + bx$, but not listed in the table, the substitution $u = \frac{a + bx}{x}$ ay prove helpful.

3. $$\int (a + bx)^n\,dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \qquad (n \neq -1)$$

4. $$\int x(a + bx)^n\,dx$$

$$= \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}, \qquad (n \neq -1, -2)$$

25. $$\int x^2(a + bx)^n\,dx = \frac{1}{b^3}\left[\frac{(a + bx)^{n+3}}{n+3} - 2a\frac{(a + bx)^{n+2}}{n+2} + a^2\frac{(a + bx)^{n+1}}{n+1}\right]$$

INTEGRALS (Continued)

26. $$\int x^m(a+bx)^n\,dx = \begin{cases} \dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\displaystyle\int x^m(a+bx)^{n-1}\,dx \\ \text{or} \\ \dfrac{1}{a(n+1)}\Big[-x^{m+1}(a+bx)^{n+1} \\ \qquad + (m+n+2)\displaystyle\int x^m(a+bx)^{n+1}\,dx\Big] \\ \text{or} \\ \dfrac{1}{b(m+n+1)}\Big[x^m(a+bx)^{n+1} - ma\displaystyle\int x^{m-1}(a+bx)^n\,dx\Big] \end{cases}$$

27. $$\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx)$$

28. $$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

29. $$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}$$

first terms; $a_2 = S_2 - S_1$, plug in for n

30. $$\int \frac{x\,dx}{a+bx} = \begin{cases} \dfrac{1}{b^2}[a+bx-a\log(a+bx)] \\ \text{or} \\ \dfrac{x}{b} - \dfrac{a}{b^2}\log(a+bx) \end{cases}$$

31. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right]$$

32. $$\int \frac{x\,dx}{(a+bx)^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2$$

33. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right]$$

34. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right]$$

35. $$\int \frac{x^2\,dx}{(a+bx)^3} = \frac{1}{b^3}\left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$$

36. $$\int \frac{x^2\,dx}{(a+bx)^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2, 3$$

INTEGRALS (Continued)

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left[\frac{1}{2}\left(\frac{2a+bx}{a+bx}\right)^2 + \log\frac{x}{a+bx}\right]$$

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3}\log\frac{x}{a+bx}$$

$$\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}$$

FORMS CONTAINING $c^2 \pm x^2$, $x^2 - c^2$

3. $$\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}$$

4. $$\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}, \qquad (c^2 > x^2)$$

5. $$\int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}, \qquad (x^2 > c^2)$$

6. $$\int \frac{x\,dx}{c^2 \pm x^2} = \pm\frac{1}{2}\log(c^2 \pm x^2)$$

7. $$\int \frac{x\,dx}{(c^2 \pm x^2)^{n+1}} = \mp\frac{1}{2n(c^2 \pm x^2)^n}$$

48. $$\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)}\left[\frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3)\int\frac{dx}{(c^2 \pm x^2)^{n-1}}\right]$$

49. $$\int \frac{dx}{(x^2 - c^2)^n} = \frac{1}{2c^2(n-1)}\left[-\frac{x}{(x^2 - c^2)^{n-1}} - (2n-3)\int\frac{dx}{(x^2 - c^2)^{n-1}}\right]$$

50. $$\int \frac{x\,dx}{x^2-c^2} = \frac{1}{2}\log(x^2-c^2)$$

51. $$\int \frac{x\,dx}{(x^2-c^2)^{n+1}} = -\frac{1}{2n(x^2-c^2)^n}$$

INTEGRALS (Continued)

FORMS CONTAINING $a + bx$ and $c + dx$

$u = a + bx, \qquad v = c + dx, \qquad k = ad - bc$

If $k = 0$, then $v = \frac{c}{a}u$

52. $$\int \frac{dx}{u \cdot v} = \frac{1}{k} \cdot \log\left(\frac{v}{u}\right)$$

53. $$\int \frac{x\,dx}{u \cdot v} = \frac{1}{k}\left[\frac{a}{b}\log(u) - \frac{c}{d}\log(v)\right]$$

54. $$\int \frac{dx}{u^2 \cdot v} = \frac{1}{k}\left(\frac{1}{u} + \frac{d}{k}\log\frac{v}{u}\right)$$

55. $$\int \frac{x\,dx}{u^2 \cdot v} = \frac{-a}{bku} - \frac{c}{k^2}\log\frac{v}{u}$$

56. $$\int \frac{x^2\,dx}{u^2 \cdot v} = \frac{a^2}{b^2ku} + \frac{1}{k^2}\left[\frac{c^2}{d}\log(v) + \frac{a(k - bc)}{b^2}\log(u)\right]$$

57. $$\int \frac{dx}{u^n \cdot v^m} = \frac{1}{k(m-1)}\left[\frac{-1}{u^{n-1} \cdot v^{m-1}} - (m + n - 2)b\int \frac{dx}{u^n \cdot v^{m-1}}\right]$$

58. $$\int \frac{u}{v}\,dx = \frac{bx}{d} + \frac{k}{d^2}\log(v)$$

59. $$\int \frac{u^m\,dx}{v^n} = \begin{cases} \dfrac{-1}{k(n-1)}\left[\dfrac{u^{m+1}}{v^{n-1}} + b(n - m - 2)\displaystyle\int \dfrac{u^m}{v^{n-1}}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n-m-1)}\left[\dfrac{u^m}{v^{n-1}} + mk\displaystyle\int \dfrac{u^{m-1}}{v^n}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n-1)}\left[\dfrac{u^m}{v^{n-1}} - mb\displaystyle\int \dfrac{u^{m-1}}{v^{n-1}}\,dx\right] \end{cases}$$

FORMS CONTAINING $(a + bx^n)$

60. $$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a}, \qquad (ab > 0)$$

61. $$\int \frac{dx}{a + bx^2} = \begin{cases} \dfrac{1}{2\sqrt{-ab}}\log\dfrac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & (ab < 0) \\ \text{or} \\ \dfrac{1}{\sqrt{-ab}}\tanh^{-1}\dfrac{x\sqrt{-ab}}{a}, & (ab < 0) \end{cases}$$

INTEGRALS (Continued)

$$= \frac{1}{ab}\tan^{-1}\frac{bx}{a}$$

$$= \frac{1}{2b}\log(a + bx^2)$$

$$= \frac{x}{b} - \frac{a}{b}\int\frac{dx}{a + bx^2}$$

$$x^2)^2 = \frac{x}{2a(a + bx^2)} + \frac{1}{2a}\int\frac{dx}{a + bx^2}$$

$$b^2x^2 = \frac{1}{2ab}\log\frac{a + bx}{a - bx}$$

$$\frac{dx}{+ bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2ma}\dfrac{x}{(a + bx^2)^m} + \dfrac{2m - 1}{2ma}\displaystyle\int\frac{dx}{(a + bx^2)^m} \\ \text{or} \\ \dfrac{(2m)!}{(m!)^2}\left[\dfrac{x}{2a}\displaystyle\sum_{r=1}^{m}\frac{r!(r-1)!}{(4a)^{m-r}(2r)!(a + bx^2)^r} + \frac{1}{(4a)^m}\int\frac{dx}{a + bx^2}\right] \end{cases}$$

$$\frac{x\,dx}{a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}$$

$$\frac{x^2\,dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb}\int\frac{dx}{(a + bx^2)^m}$$

$$\int\frac{dx}{x(a + bx^2)} = \frac{1}{2a}\log\frac{x^2}{a + bx^2}$$

$$\int\frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int\frac{dx}{a + bx^2}$$

2. $$\int\frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2am(a + bx^2)^m} + \dfrac{1}{a}\displaystyle\int\frac{dx}{x(a + bx^2)^m} \\ \text{or} \\ \dfrac{1}{2a^{m+1}}\left[\displaystyle\sum_{r=1}^{m}\frac{a^r}{r(a + bx^2)^r} + \log\frac{x^2}{a + bx^2}\right] \end{cases}$$

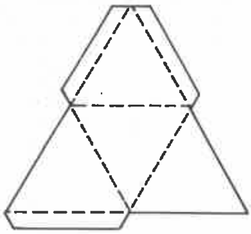
73. $$\int\frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a}\int\frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a}\int\frac{dx}{(a + bx^2)^{m+1}}$$

74. $$\int\frac{dx}{a + bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\frac{(k + x)^3}{a + bx^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

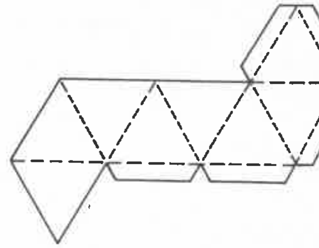
75. $$\int\frac{x\,dx}{a + bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\frac{a + bx^3}{(k + x)^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

PATTERNS OF REGULAR POLYHEDRA

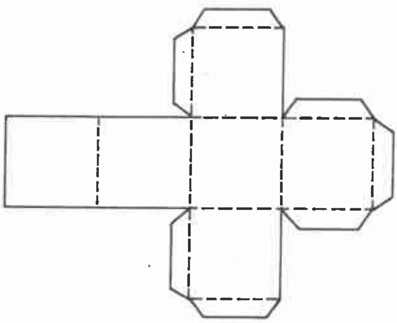
Tetrahedron



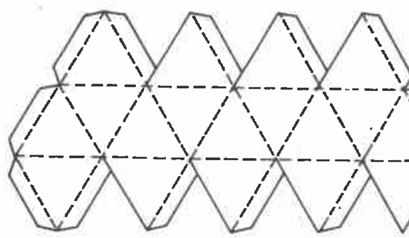
Octahedron



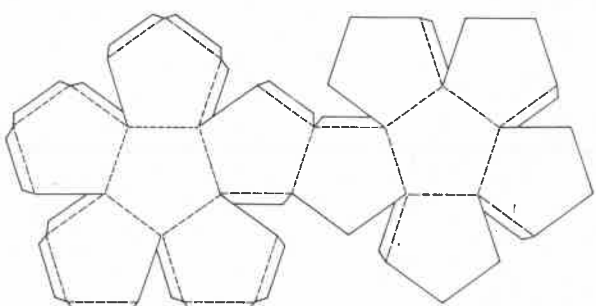
Hexahedron, or Cube



Icosahedron



Dodecahedron



To graph $\int_0^2 \int_{2x}^{x^2} (x-2y)\,dy\,dx$

Use ⇒ $y = 2x$ and $y = x^2$ for $x = 0$ to $x = 2$

if want to switch to $\int\int dx\,dy$ then $x = y/2$ and

The limits on y must be values of y that corre

to $x = 0$ and $x = 2$.

$= \int^0 \int^{y/2}$

CALCULUS

Derivatives *

the following formulas u, v, w represent functions of x, while a, c, n represent fixed real bers. All arguments in the trigonometric functions are measured in radians, and all inverse nometric and hyperbolic functions represent principal values.

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(au) = a\frac{du}{dx}$$

$$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$ and so on to n factors

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}}\left(nv\frac{du}{dx} - mu\frac{dv}{dx}\right)$$

$$\frac{d}{dx}(u^n v^m) = u^{n-1}v^{m-1}\left(nv\frac{du}{dx} + mu\frac{dv}{dx}\right)$$

$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$$

Max + Min

(1) Find Z_x and Z_y + set them both equal to zero. Solve simultaneously to get number values for $x + y$. See if there is a max or min of the function it occurs at this pt. of $x + y$.

(2) Find Z_{xx} and Z_{yy} + Z_{xy} ← This is the second derivative of (if taken) w/r to x.

(3) Plug into

$Z_{xx}Z_{yy} - (Z_{xy})^2$

(a) and if >0 and $Z_{xx} + Z_{yy}$ are neg = max

(b) if >0 but $Z_{xx} + Z_{yy}$ are positive = min

(c) if <0 it is a "critical value which is neither a max or min

(d) if $=0$ the test fails to give desired info

et $y = f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any ue x in their common domain. The differential for the function at such a value x is ordingly defined as

$$dy = d[f(x)] = \frac{dy}{dx}\,dx = \frac{d[f(x)]}{dx}\,dx = f'(x)dx$$

ch derivative formula has an associated differential formula. For example, formula 6 above the differential formula

) ntl osilate $N=1 \Rightarrow N=-1$

mis Sw D.M E. SEQ

$$d(uvw) = uv\,dw + vw\,du + uw\,dv$$

$L+1 \to \pm\infty$ ∴ Diverges

DERIVATIVES (Continued)

15. $\frac{d^2}{dx^2}[f(u)] = \frac{df(u)}{du} \cdot \frac{d^2u}{dx^2} + \frac{d^2f(u)}{du^2} \cdot \left(\frac{du}{dx}\right)^2$

16. $\frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \binom{n}{2} \frac{d^2v}{dx^2} \frac{d^{n-2}u}{dx^{n-2}}$

$+ \cdots + \binom{n}{k} \frac{d^kv}{dx^k} \frac{d^{n-k}u}{dx^{n-k}} + \cdots + \binom{n}{n} u$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ the binomial coefficient, n non-negative integer and $\binom{n}{0} = 1.$

17. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$ if $\frac{dx}{du} \neq 0$

18. $\frac{d}{dx}(\log_a u) = (\log_a e)\frac{1}{u}\frac{du}{dx}$

19. $\frac{d}{dx}(\log_e u) = \frac{1}{u}\frac{du}{dx}$

20. $\frac{d}{dx}(a^u) = a^u(\log_e a)\frac{du}{dx}$

21. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

22. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + (\log_e u)u^v\frac{dv}{dx}$

23. $\frac{d}{dx}(\sin u) = \frac{du}{dx}(\cos u)$

24. $\frac{d}{dx}(\cos u) = -\frac{du}{dx}(\sin u)$

25. $\frac{d}{dx}(\tan u) = \frac{du}{dx}(\sec^2 u)$

26. $\frac{d}{dx}(\cot u) = -\frac{du}{dx}(\csc^2 u)$

27. $\frac{d}{dx}(\sec u) = \frac{du}{dx} \sec u \cdot \tan u$

28. $\frac{d}{dx}(\csc u) = -\frac{du}{dx} \csc u \cdot \cot u$

29. $\frac{d}{dx}(\text{vers } u) = \frac{du}{dx} \sin u$

30. $\frac{d}{dx}(\text{arc} \sin u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$, $\left(-\frac{\pi}{2} \leq \text{arc} \sin u \leq \frac{\pi}{2}\right)$

(A) Partial Derivati

① $du = u_x dx + u_y dy + u_z dz$

② if $V = \pi r^2 h$ (tuna can Prob.)

$dV = V_r dr + V_h dh = 2\pi rh\, dr + \pi$

③ Relative error:

$\frac{\text{Approximate value}}{\text{actual value}} = \frac{dV}{V}$

(B) TOTAL DERIVATIVES FROM Partia

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$

(C) Diff of Implicit Function

$\frac{dy}{dx} = -\left(\frac{\partial\phi/\partial x}{\partial\phi/\partial y}\right)$ if $\frac{\partial\phi}{\partial y} \neq 0$

L'Hopital's Rule 0/0

DERIVATIVES (Continued)

$$\frac{d}{dx}(\text{arc cos } u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad (0 \le \text{arc cos } u \le \pi)$$

$$\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1+u^2}\frac{du}{dx}, \qquad \left(-\frac{\pi}{2} < \text{arc tan } u < \frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc cot } u) = -\frac{1}{1+u^2}\frac{du}{dx}, \qquad (0 \le \text{arc cot } u \le \pi)$$

$$\frac{d}{dx}(\text{arc sec } u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \qquad \left(0 \le \text{arc sec } u < \frac{\pi}{2}, -\pi \le \text{arc sec } u < -\frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc csc } u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \qquad \left(0 < \text{arc csc } u \le \frac{\pi}{2}, -\pi < \text{arc csc } u \le -\frac{\pi}{2}\right)$$

$$\frac{d}{dx}(\text{arc vers } u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}, \qquad (0 \le \text{arc vers } u \le \pi)$$

$$\frac{d}{dx}(\sinh u) = \frac{du}{dx}(\cosh u)$$

$$\frac{d}{dx}(\cosh u) = \frac{du}{dx}(\sinh u)$$

$$\frac{d}{dx}(\tanh u) = \frac{du}{dx}(\text{sech}^2 u)$$

$$\frac{d}{dx}(\coth u) = -\frac{du}{dx}(\text{csch}^2 u)$$

$$\frac{d}{dx}(\text{sech } u) = -\frac{du}{dx}(\text{sech } u \cdot \tanh u)$$

$$\frac{d}{dx}(\text{csch } u) = -\frac{du}{dx}(\text{csch } u \cdot \coth u)$$

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2+1})] = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2-1})] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad (u > 1, \cosh^{-1} u > 0)$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \qquad (u^2 < 1)$$

$$\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \qquad (u^2 > 1)$$

$$\frac{d}{dx}(\text{sech}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad (0 < u < 1, \text{sech}^{-1} u > 0)$$

$$\frac{d}{dx}(\text{csch}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

49. $\dfrac{d}{dq}\displaystyle\int_p^q f(x)\,dx = f(q),$ [p constant]

50. $\dfrac{d}{dp}\displaystyle\int_p^q f(x)\,dx = -f(p),$ [q constant]

51. $\dfrac{d}{da}\displaystyle\int_p^q f(x,a)\,dx = \int_p^q \frac{\partial}{\partial a}[f(x,a)]\,dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$

① PARTIAL FRACTIONS/RATIONAL FRACTIONS CASE

$$\frac{x+C}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

DEFN OF BASE e

$\lim_{x\to 0}(1+x)^{1/x}$ $\quad \ln y = \lim_{x\to 0}$ ...

$= \lim \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

$= \frac{1}{1} = 1 \quad \ln y = 1$

$= e$

② Area Trapezoidal Rule:

$$\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)\right]$$

$$\Delta x = \frac{x_n - x_0}{n} = \frac{\text{upper limit} - \text{lower limit}}{n_{(\text{given})}}$$

③ Rectangular Area

$$= \int (f_1(x) - f_2(x))\,dx$$

$0 \cdot \infty$: $\lim_{x\to \pi/2} \sec 3x \cos 5x = \lim_{x\to\pi/2}$ ...

$= \lim_{x\to\pi/2} \frac{+5\sin 5x}{+3\sin 3x} = -$ ...

④ Polar Area:

$$= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$

⑤ $Vol_{disk} = \pi \int r^2\,dh$

⑥ $Vol_{by\ shells} = 2\pi \int r \cdot h \cdot dr$

⑦ $s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ or $\sqrt{1 + \frac{dx}{dy}}\,dy$ Arc length $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (\ldots}$

⑧ $S = 2\pi \int_{x_1}^{x_2} r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ Surface Area

INTEGRATION

: following is a brief discussion of some integration techniques. A more complete dis-
n can be found in a number of good text books. However, the purpose of this introduction
ply to discuss a few of the important techniques which may be used, in conjunction with
tegral table which follows, to integrate particular functions.

matter how extensive the integral table, it is a fairly uncommon occurrence to find
table the exact integral desired. Usually some form of transformation will have to be
. The simplest type of transformation, and yet the most general, is substitution. Simple
of substitution, such as $y = ax$, are employed almost unconsciously by experienced
of integral tables. Other substitutions may require more thought. In some sections of
bles, appropriate substitutions are suggested for integrals which are similar to, but not
ly like, integrals in the table. Finding the right substitution is largely a matter of intuition
xperience.

eral precautions must be observed when using substitutions:

: sure to make the substitution in the dx term, as well as everywhere else in the integral.
: sure that the function substituted is one-to-one and continuous. If this is not the case,
e integral must be restricted in such a way as to make it true. See the example following.
ith definite integrals, the limits should also be expressed in terms of the new dependent
riable. With indefinite integrals, it is necessary to perform the reverse substitution to
tain the answer in terms of the original independent variable. This may also be done
r definite integrals, but it is usually easier to change the limits.

ample:

$$\int \frac{x^4}{\sqrt{a^2 - x^2}}\,dx$$

ere we make the substitution $x = |a| \sin\theta$. Then $dx = |a| \cos\theta\, d\theta$, and

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = |a|\sqrt{1 - \sin^2\theta} = |a \cos\theta|$$

ice the absolute value signs. It is very important to keep in mind that a square root radical
ys denotes the positive square root, and to assure the sign is always kept positive. Thus
$^2 = |x|$. Failure to observe this is a common cause of errors in integration.

otice also that the indicated substitution is not a one-to-one function, that is, it does not
e a unique inverse. Thus we must restrict the range of θ in such a way as to make the
tion one-to-one. Fortunately, this is easily done by solving for θ

$$\theta = \sin^{-1} \frac{x}{|a|}$$

restricting the inverse sine to the principal values, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

hus the integral becomes

$$\int \frac{a^4 \sin^4\theta |a| \cos\theta\, d\theta}{|a||\cos\theta|}$$

w, however, in the range of values chosen for θ, $\cos\theta$ is always positive. Thus we may remove
absolute value signs from $\cos\theta$ in the denominator. (This is one of the reasons that the
ncipal values of the inverse trigonometric functions are defined as they are.)

Then the $\cos\theta$ terms cancel, and the integral becomes

$$a^4 \int \sin^4 \theta \, d\theta$$

By application of integral formulas 299 and 296, we integrate this to

$$-a^4 \frac{\sin^3 \theta \cos \theta}{4} - \frac{3a^4}{8} \cos \theta \sin \theta + \frac{3a^4}{8} \theta + C$$

We now must perform the inverse substitution to get the result in terms of x. We have

$$\theta = \sin^{-1} \frac{x}{|a|}$$

$$\sin \theta = \frac{x}{|a|}$$

Then

$$\cos \theta = \pm\sqrt{1 - \sin^2 \theta} = \pm\sqrt{1 - \frac{x^2}{a^2}} = \pm\frac{\sqrt{a^2 - x^2}}{|a|}.$$

Because of the previously mentioned fact that $\cos\theta$ is positive, we may omit the $\pm$ sig[n]. The reverse substitution then produces the final answer

$$\int \frac{x^4}{\sqrt{a^2 - x^2}} dx = -\tfrac{1}{4}x^3\sqrt{a^2 - x^2} - \tfrac{3}{8}a^2 x\sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{|a|} + C.$$

Any rational function of x may be integrated, if the denominator is factored into linea[r] and irreducible quadratic factors. The function may then be broken into partial fraction[s] and the individual partial fractions integrated by use of the appropriate formula from th[e] integral table. See the section on partial fractions for further information.

Many integrals may be reduced to rational functions by proper substitutions. For example

$$z = \tan \frac{x}{2}$$

will reduce any rational function of the six trigonometric functions of x to a rational functio[n] of z. (Frequently there are other substitutions which are simpler to use, but this one will alway[s] work. See integral formula number 484.)

Any rational function of x and $\sqrt{ax + b}$ may be reduced to a rational function of z by making the substitution

$$z = \sqrt{ax + b}.$$

Other likely substitutions will be suggested by looking at the form of the integrand.

The other main method of transforming integrals is integration by parts. This involves applying formula number 5 or 6 in the accompanying integral table. The critical factor in this method is the choice of the functions u and v. In order for the method to be successful, $v = \int dv$ and $\int v\, du$ must be easier to integrate than the original integral. Again, this choice is largely a matter of intuition and experience.

Example:

$$\int x \sin x \, dx$$

Two obvious choices are $u = x$, $dv = \sin x\, dx$, or $u = \sin x$, $dv = x\, dx$. Since a preliminary mental calculation indicates that $\int v\, du$ in the second choice would be more, rather than less, nplicated than the original integral (it would contain x^2), we use the first choice.

$$u = x \qquad du = dx$$

$$dv = \sin x\,dx \qquad v = -\cos x$$

$$\int x \sin x\,dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\,dx$$

$$= \sin x - x\cos x$$

course, this result could have been obtained directly from the integral table, but it provides imple example of the method. In more complicated examples the choice of u and v may t be so obvious, and several different choices may have to be tried. Of course, there is no arantee that any of them will work.

Integration by parts may be applied more than once, or combined with substitution. A rly common case is illustrated by the following example.

Example:

$$\int e^x \sin x\,dx$$

Let

$$u = e^x \qquad \text{Then} \quad du = e^x\,dx$$

$$dv = \sin x\,dx \qquad v = -\cos x$$

$$\int e^x \sin x\,dx = \int u\,dv = uv - \int v\,du = -e^x\cos x + \int e^x \cos x\,dx$$

In this latter integral,

$$\text{let} \quad u = e^x \qquad \text{Then} \quad du = e^x\,dx$$

$$dv = \cos x\,dx. \qquad v = \sin x$$

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx = -e^x\cos x + \int u\,dv$$

$$= -e^x \cos x + uv - \int v\,du$$

$$= -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

[his looks as if a circular transformation has taken place, since we are back at the same ntegral we started from. However, the above equation can be solved algebraically for the equired integral:

$$\int e^x \sin x\,dx = \tfrac{1}{2}(e^x \sin x - e^x \cos x)$$

In the second integration by parts, if the parts had been chosen as $u = \cos x$, $dv = e^x\,dx$, we would indeed have made a circular transformation, and returned to the starting place. In general, when doing repeated integration by parts, one should never choose the function u at any stage to be the same as the function v at the previous stage, or a constant times the previous v.

The following rule is called the extended rule for integration by parts. It is the result of $n + 1$ successive applications of integration by parts.

If

$$g_1(x) = \int g(x)\,dx, \qquad g_2(x) = \int g_1(x)\,dx,$$

$$g_3(x) = \int g_2(x)\,dx, \ldots, g_m(x) = \int g_{m-1}(x)\,dx, \ldots,$$

then

$$\int f(x)\cdot g(x)\,dx = f(x)\cdot g_1(x) - f'(x)\cdot g_2(x) + f''(x)\cdot g_3(x) - + \cdots$$
$$+ (-1)^n f^{(n)}(x) g_{n+1}(x) + (-1)^{n+1} \int f^{(n+1)}(x) g_{n+1}(x)\,dx$$

A useful special case of the above rule is when $f(x)$ is a polynomial of degree n. The $f^{(n+1)}(x) = 0$, and

$$\int f(x)\cdot g(x)\,dx = f(x)\cdot g_1(x) - f'(x)\cdot g_2(x) + f''(x)\cdot g_3(x) - + \cdots + (-1)^n f^{(n)}(x) g_{n+1}(x) +$$

Example:

If $f(x) = x^2$, $g(x) = \sin x$

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Another application of this formula occurs if

$$f''(x) = af(x) \quad \text{and} \quad g''(x) = bg(x),$$

where a and b are unequal constants. In this case, by a process similar to that used in th above example for $\int e^x \sin x\,dx$, we get the formula

$$\int f(x)g(x)\,dx = \frac{f(x)\cdot g'(x) - f'(x)\cdot g(x)}{b - a} + C$$

This formula could have been used in the example mentioned. Here is another example.

Example:

If $f(x) = e^{2x}$, $g(x) = \sin 3x$, then $a = 4$, $b = -9$, and

$$\int e^{2x} \sin 3x\,dx = \frac{3\,e^{2x}\cos 3x - 2\,e^{2x}\sin 3x}{-9 - 4} + C = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

The following additional points should be observed when using this table.

1. A constant of integration is to be supplied with the answers for indefinite integrals.
2. Logarithmic expressions are to base $e = 2.71828\ldots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.
3. All angles are measured in radians, and inverse trigonometric and hyperbolic functions represent principal values, unless otherwise indicated.
4. If the application of a formula produces either a zero denominator or the square root of a negative number in the result, there is usually available another form of the answer which avoids this difficulty. In many of the results, the excluded values are specified, but when such are omitted it is presumed that one can tell what these should be, especially when difficulties of the type herein mentioned are obtained.
5. When inverse trigonometric functions occur in the integrals, be sure that any replacements made for them are strictly in accordance with the rules for such functions. This causes

tle difficulty when the argument of the inverse trigonometric function is positive, since en all angles involved are in the first quadrant. However, if the argument is negative, »ecial care must be used. Thus if $u > 0$,

$$\sin^{-1} u = \cos^{-1}\sqrt{1-u^2} = \csc^{-1}\frac{1}{u}, \text{ etc.}$$

lowever, if $u < 0$,

$$\sin^{-1} u = -\cos^{-1}\sqrt{1-u^2} = -\pi - \csc^{-1}\frac{1}{u}, \text{ etc.}$$

ee the section on inverse trigonometric functions for a full treatment of the allowable ubstitutions.

n integrals 340–345 and some others, the right side includes expressions of the form

$$A \tan^{-1}[B + C \tan f(x)].$$

n these formulas, the $\tan^{-1}$ does not necessarily represent the principal value. Instead of ılways employing the principal branch of the inverse tangent function, one must instead ıse that branch of the inverse tangent function upon which $f(x)$ lies for any particular :hoice of x.

:xample:

$$\int_0^{4\pi} \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\Bigg]_0^{4\pi}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2\tan 2\pi + 1}{\sqrt{3}} - \tan^{-1}\frac{2\tan 0 + 1}{\sqrt{3}}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{13\pi}{6} - \frac{\pi}{6}\right] = \frac{4\pi}{\sqrt{3}} = \frac{4\sqrt{3}\pi}{3}$$

:re

$$\tan^{-1}\frac{2\tan 2\pi + 1}{\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{13\pi}{6},$$

nce $f(x) = 2\pi$; and

$$\tan^{-1}\frac{2\tan 0 + 1}{\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6},$$

nce $f(x) = 0$.

B_n and E_n where used in Integrals represents the Bernoulli and Euler numbers as defined in the tables contained from pages 408 to 414.

Laplace 1st translation theorem $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a} = F(s-a)$

$\mathcal{L}\{e^{5t}t^3\} = \mathcal{L}\{t^3\}_{s\to s-5} = \frac{3!}{s^4}\Big|_{s\to s-5} = \frac{6}{(s-5)^4}$

$\mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{\cos 4t\}_{s\to s+2} = \frac{s}{s^2+16}\Big|_{s\to s+2}$

INTEGRALS

ELEMENTARY FORMS

$e^0 = 1$

1. $\int a\,dx = ax$

2. $\int a \cdot f(x)\,dx = a\int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \frac{\phi(y)}{y'}\,dy,$ where $y' = \frac{dy}{dx}$

4. $\int (u + v)\,dx = \int u\,dx + \int v\,dx,$ where u and v are any functions of x

$F(t) = e^{t+7}$

5. $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du$

6. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

7. $\int x^n\,dx = \frac{x^{n+1}}{n+1},$ except $n = -1$

8. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x),$ $(df(x) = f'(x)\,dx)$

9. $\int \frac{dx}{x} = \log x$

10. $\int \frac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)},$ $(df(x) = f'(x)\,dx)$

$\int \ln|x|\,dx = x\ln\ldots$

11. $\int e^x\,dx = e^x$

12. $\int e^{ax}\,dx = e^{ax}/a$

$\int x^n \ln|x|\,dx = \frac{x^{n+1}}{n+1}\left(\ln|x| - \frac{1}{n+1}\right)$

13. $\int b^{ax}\,dx = \frac{b^{ax}}{a\log b},$ $(b > 0)$

14. $\int \log x\,dx = x\log x - x$

15. $\int a^x \log a\,dx = a^x,$ $(a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int x^2 \ln x\,dx = \frac{x^3(6\ln|x| - 1)}{9}$

INTEGRALS (Continued)

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{a+x}{a-x}, \quad (a^2 > x^2) \end{cases}$$

$$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a}\coth^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{x-a}{x+a}, \quad (x^2 > a^2) \end{cases}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1}\frac{x}{|a|} \\ \text{or} \\ -\cos^{-1}\frac{x}{|a|}, \quad (a^2 > x^2) \end{cases}$$

. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

FORMS CONTAINING $(a + bx)$

For forms containing $a + bx$, but not listed in the table, the substitution $u = \frac{a + bx}{x}$ ay prove helpful.

3. $$\int (a + bx)^n\,dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \qquad (n \neq -1)$$

4. $$\int x(a + bx)^n\,dx$$
$$= \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}, \qquad (n \neq -1, -2)$$

25. $$\int x^2(a + bx)^n\,dx = \frac{1}{b^3}\left[\frac{(a + bx)^{n+3}}{n+3} - 2a\frac{(a + bx)^{n+2}}{n+2} + a^2\frac{(a + bx)^{n+1}}{n+1}\right]$$

26. $$\int x^m(a+bx)^n\,dx = \begin{cases} \dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\displaystyle\int x^m(a+bx)^{n-1}\,dx \\ \text{or} \\ \dfrac{1}{a(n+1)}\left[-x^{m+1}(a+bx)^{n+1} + (m+n+2)\displaystyle\int x^m(a+bx)^{n+1}\,dx\right] \\ \text{or} \\ \dfrac{1}{b(m+n+1)}\left[x^m(a+bx)^{n+1} - ma\displaystyle\int x^{m-1}(a+bx)^n\,dx\right] \end{cases}$$

27. $$\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx)$$

28. $$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

29. $$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}$$

30. $$\int \frac{x\,dx}{a+bx} = \begin{cases} \dfrac{1}{b^2}[a+bx-a\log(a+bx)] \\ \text{or} \\ \dfrac{x}{b} - \dfrac{a}{b^2}\log(a+bx) \end{cases}$$

31. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right]$$

32. $$\int \frac{x\,dx}{(a+bx)^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2$$

33. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right]$$

34. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right]$$

35. $$\int \frac{x^2\,dx}{(a+bx)^3} = \frac{1}{b^3}\left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$$

36. $$\int \frac{x^2\,dx}{(a+bx)^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2, 3$$

INTEGRALS (Continued)

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left[\frac{1}{2}\left(\frac{2a+bx}{a+bx}\right)^2 + \log\frac{x}{a+bx}\right]$$

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$$

$$\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3}\log\frac{x}{a+bx}$$

$$\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}$$

FORMS CONTAINING $c^2 \pm x^2$, $x^2 - c^2$

3. $$\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}$$

4. $$\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}, \qquad (c^2 > x^2)$$

5. $$\int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}, \qquad (x^2 > c^2)$$

6. $$\int \frac{x\,dx}{c^2 \pm x^2} = \pm\frac{1}{2}\log(c^2 \pm x^2)$$

7. $$\int \frac{x\,dx}{(c^2 \pm x^2)^{n+1}} = \mp\frac{1}{2n(c^2 \pm x^2)^n}$$

48. $$\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)}\left[\frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{dx}{(c^2 \pm x^2)^{n-1}}\right]$$

49. $$\int \frac{dx}{(x^2-c^2)^n} = \frac{1}{2c^2(n-1)}\left[-\frac{x}{(x^2-c^2)^{n-1}} - (2n-3)\int \frac{dx}{(x^2-c^2)^{n-1}}\right]$$

50. $$\int \frac{x\,dx}{x^2-c^2} = \frac{1}{2}\log(x^2-c^2)$$

51. $$\int \frac{x\,dx}{(x^2-c^2)^{n+1}} = -\frac{1}{2n(x^2-c^2)^n}$$

INTEGRALS (Continued)

FORMS CONTAINING $a + bx$

$u = a + bx, \qquad v = c + dx, \qquad$ and $c + dx$ $= ad - bc$

If $k = 0$, then $v = \frac{c}{a}u$

52. $$\int \frac{dx}{u \cdot v} = \frac{1}{k} \cdot \log\left(\frac{v}{u}\right)$$

53. $$\int \frac{x\,dx}{u \cdot v} = \frac{1}{k}\left[\frac{a}{b}\log(u) - \frac{c}{d}\log(v)\right]$$

54. $$\int \frac{dx}{u^2 \cdot v} = \frac{1}{k}\left(\frac{1}{u} + \frac{d}{k}\log\frac{v}{u}\right)$$

55. $$\int \frac{x\,dx}{u^2 \cdot v} = \frac{-a}{bku} - \frac{c}{k^2}\log\frac{v}{u}$$

56. $$\int \frac{x^2\,dx}{u^2 \cdot v} = \frac{a^2}{b^2ku} + \frac{1}{k^2}\left[\frac{c^2}{d}\log(v) + \frac{a(k - bc)}{b^2}\log(u)\right]$$

57. $$\int \frac{dx}{u^n \cdot v^m} = \frac{1}{k(m-1)}\left[\frac{-1}{u^{n-1} \cdot v^{m-1}} - (m + n - 2)b \int \frac{dx}{u^n \cdot v^{m-1}}\right]$$

58. $$\int \frac{u}{v}\,dx = \frac{bx}{d} + \frac{k}{d^2}\log(v)$$

59. $$\int \frac{u^m\,dx}{v^n} = \begin{cases} \dfrac{-1}{k(n-1)}\left[\dfrac{u^{m+1}}{v^{n-1}} + b(n - m - 2)\displaystyle\int \dfrac{u^m}{v^{n-1}}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n - m - 1)}\left[\dfrac{u^m}{v^{n-1}} + mk\displaystyle\int \dfrac{u^{m-1}}{v^n}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n-1)}\left[\dfrac{u^m}{v^{n-1}} - mb\displaystyle\int \dfrac{u^{m-1}}{v^{n-1}}\,dx\right] \end{cases}$$

FORMS CONTAINING $(a + bx^n)$

60. $$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a}, \qquad (ab > 0)$$

61. $$\int \frac{dx}{a + bx^2} = \begin{cases} \dfrac{1}{2\sqrt{-ab}}\log\dfrac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & (ab < 0) \\ \text{or} \\ \dfrac{1}{\sqrt{-ab}}\tanh^{-1}\dfrac{x\sqrt{-ab}}{a}, & (ab < 0) \end{cases}$$

INTEGRALS (Continued)

$$\frac{dx}{+ b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a}$$

$$\frac{x\,dx}{+ bx^2} = \frac{1}{2b}\log(a + bx^2)$$

$$\frac{x^2\,dx}{+ bx^2} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{a + bx^2}$$

$$\frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a}\int\frac{dx}{a + bx^2}$$

$$\frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab}\log\frac{a + bx}{a - bx}$$

$$\int\frac{dx}{(a + bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2ma}\dfrac{x}{(a + bx^2)^m} + \dfrac{2m - 1}{2ma}\displaystyle\int\frac{dx}{(a + bx^2)^m} \\ \text{or} \\ \dfrac{(2m)!}{(m!)^2}\left[\dfrac{x}{2a}\displaystyle\sum_{r=1}^{m}\frac{r!(r - 1)!}{(4a)^{m-r}(2r)!(a + bx^2)^r} + \frac{1}{(4a)^m}\int\frac{dx}{a + bx^2}\right] \end{cases}$$

$$\int\frac{x\,dx}{(a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}$$

$$\int\frac{x^2\,dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb}\int\frac{dx}{(a + bx^2)^m}$$

0. $$\int\frac{dx}{x(a + bx^2)} = \frac{1}{2a}\log\frac{x^2}{a + bx^2}$$

'1. $$\int\frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int\frac{dx}{a + bx^2}$$

72. $$\int\frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2am(a + bx^2)^m} + \dfrac{1}{a}\displaystyle\int\frac{dx}{x(a + bx^2)^m} \\ \text{or} \\ \dfrac{1}{2a^{m+1}}\left[\displaystyle\sum_{r=1}^{m}\frac{a^r}{r(a + bx^2)^r} + \log\frac{x^2}{a + bx^2}\right] \end{cases}$$

73. $$\int\frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a}\int\frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a}\int\frac{dx}{(a + bx^2)^{m+1}}$$

74. $$\int\frac{dx}{a + bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\frac{(k + x)^3}{a + bx^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

75. $$\int\frac{x\,dx}{a + bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\frac{a + bx^3}{(k + x)^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

76. $\int \frac{x^2\,dx}{a + bx^3} = \frac{1}{3b} \log (a +$

77. $\int \frac{dx}{a + bx^4} = \frac{k}{2a}\left[\frac{1}{2}\log\frac{x^2 +}{x^2 -}\right.$

78. $\int \frac{dx}{a + b} = \frac{k}{2a}\left[\frac{1}{2}\log\frac{x+k}{x-k} + \tan^{-1}\right.$

79. $\int \frac{x\,dx}{a + bx^4} = \frac{1}{2bk}\tan^{-1}\frac{x^2}{k}, \quad (ab > 0, k$

80. $\int \frac{x\,dx}{a + bx^4} = \frac{1}{4bk}\log\frac{x^2 - k}{x^2 + k}, \quad (ab < 0, k =$

81. $\int \frac{x^2\,dx}{a + bx^4} = \frac{1}{4bk}\left[\frac{1}{2}\log\frac{x^2 - 2kx + 2k^2}{x^2 + 2kx + 2k^2} + \tan^{-1}\frac{}{2k}\right.$

82. $\int \frac{x^2\,dx}{a + bx^4} = \frac{1}{4bk}\left[\log\frac{x - k}{x + k} + 2\tan^{-1}\frac{x}{k}\right], \quad (ab < 0, k =$

83. $\int \frac{x^3\,dx}{a + bx^4} = \frac{1}{4b}\log (a + bx^4)$

84. $\int \frac{dx}{x(a + bx^n)} = \frac{1}{an}\log\frac{x^n}{a + bx^n}$

85. $\int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a}\int\frac{dx}{(a + bx^n)^m} - \frac{b}{a}\int\frac{x^n\,dx}{(a + bx^n)^{m+1}}$

86. $\int \frac{x^m\,dx}{(a + bx^n)^{p+1}} = \frac{1}{b}\int\frac{x^{m-n}\,dx}{(a + bx^n)^p} - \frac{a}{b}\int\frac{x^{m-n}\,dx}{(a + bx^n)^{p+1}}$

87. $\int \frac{dx}{x^m(a + bx^n)^{p+1}} = \frac{1}{a}\int\frac{dx}{x^m(a + bx^n)^p} - \frac{b}{a}\int\frac{dx}{x^{m-n}(a +}$

$)^{p+1}$

$n+1)\int x^{m-n}(a+bx^n)^p\,dx\Big]$

$+ bx^n)^p$

$+ anp\int x^m(a+bx^n)^{p-1}\,dx\Big]$

${}^{+1}(a+bx^n)^{p+1}$

$-(m+1+np+n)b\int x^{m+n}(a+bx^n)^p\,dx\Big]$

or

$\frac{1}{(p+1)}\Big[-x^{m+1}(a+bx^n)^{p+1}$

$+(m+1+np+n)\int x^m(a+bx^n)^{p+1}\,dx\Big]$

FORMS CONTAINING $c^3 \pm x^3$

$\pm\frac{1}{6c^2}\log\frac{(c\pm x)^3}{c^3\pm x^3}+\frac{1}{c^2\sqrt{3}}\tan^{-1}\frac{2x\mp c}{c\sqrt{3}}$

$x^3)^2 = \frac{x}{3c^3(c^3\pm x^3)}+\frac{2}{3c^3}\int\frac{dx}{c^3\pm x^3}$

$\frac{dx}{\pm x^3)^{n+1}} = \frac{1}{3nc^3}\Big[\frac{x}{(c^3\pm x^3)^n}+(3n-1)\int\frac{dx}{(c^3\pm x^3)^n}\Big]$

$\int\frac{x\,dx}{c^3\pm x^3} = \frac{1}{6c}\log\frac{c^3\pm x^3}{(c\pm x)^3}\pm\frac{1}{c\sqrt{3}}\tan^{-1}\frac{2x\mp c}{c\sqrt{3}}$

$\int\frac{x\,dx}{(c^3\pm x^3)^2} = \frac{x^2}{3c^3(c^3\pm x^3)}+\frac{1}{3c^3}\int\frac{x\,dx}{c^3\pm x^3}$

3. $\int\frac{x\,dx}{\pm x^3)^{p+1}} = \frac{1}{3nc^3}\Big[\frac{x^2}{(c^3\pm x^3)^n}+(3n-2)\int\frac{x\,dx}{(c^3\pm x^3)^n}\Big]$

$\int \ldots = \frac{1}{\ldots}\log(c^3\pm x^3)$

INTEGRALS (Continued)

96. $$\int \frac{x^2\,dx}{(c^3 \pm x^3)^{n+1}} = \mp \frac{1}{3n(c^3 \pm x^3)^n}$$

97. $$\int \frac{dx}{x(c^3 \pm x^3)} = \frac{1}{3c^3} \log \frac{x^3}{c^3 \pm x^3}$$

98. $$\int \frac{dx}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6} \log \frac{x^3}{c^3 \pm x^3}$$

99. $$\int \frac{dx}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3} \int \frac{dx}{x(c^3 \pm x^3)^n}$$

100. $$\int \frac{dx}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3} \int \frac{x\,dx}{c^3 \pm x^3}$$

101. $$\int \frac{dx}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3} \int \frac{dx}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3} \int \frac{x\,dx}{(c^3 \pm x^3)^{n+1}}$$

FORMS CONTAINING $c^4 \pm x^4$

102. $$\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^3\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 + cx\sqrt{2} + c^2}{x^2 - cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$$

103. $$\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^3} \left[\frac{1}{2} \log \frac{c + x}{c - x} + \tan^{-1} \frac{x}{c} \right]$$

104. $$\int \frac{x\,dx}{c^4 + x^4} = \frac{1}{2c^2} \tan^{-1} \frac{x^2}{c^2}$$

105. $$\int \frac{x\,dx}{c^4 - x^4} = \frac{1}{4c^2} \log \frac{c^2 + x^2}{c^2 - x^2}$$

106. $$\int \frac{x^2\,dx}{c^4 + x^4} = \frac{1}{2c\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 - cx\sqrt{2} + c^2}{x^2 + cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$$

107. $$\int \frac{x^2\,dx}{c^4 - x^4} = \frac{1}{2c} \left[\frac{1}{2} \log \frac{c + x}{c - x} - \tan^{-1} \frac{x}{c} \right]$$

108. $$\int \frac{x^3\,dx}{c^4 \pm x^4} = \pm \frac{1}{4} \log (c^4 \pm x^4)$$

FORMS CONTAINING $(a + bx + cx^2)$

$X = a + bx + cx^2$ and $q = 4ac - b^2$

If $q = 0$, then $X = c\left(x + \dfrac{b}{2c}\right)^2$, and formulas starting with 23 should be used i place of these.

109. $$\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}, \qquad (q > 0)$$

INTEGRALS (Continued)

$$\int \frac{dx}{X} = \begin{cases} \dfrac{-2}{\sqrt{-q}} \tanh^{-1} \dfrac{2cx+b}{\sqrt{-q}} \\ \text{or} \\ \dfrac{1}{\sqrt{-q}} \log \dfrac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}, \qquad (q<0) \end{cases}$$

$$\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q} \int \frac{dx}{X}$$

$$\int \frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2} + \frac{3c}{qX}\right) + \frac{6c^2}{q^2} \int \frac{dx}{X}$$

$$\int \frac{dx}{X^{n+1}} = \begin{cases} \dfrac{2cx+b}{nqX^n} + \dfrac{2(2n-1)c}{qn} \displaystyle\int \frac{dx}{X^n} \\ \text{or} \\ \dfrac{(2n)!}{(n!)^2}\left(\dfrac{c}{q}\right)^n \left[\dfrac{2cx+b}{q} \displaystyle\sum_{r=1}^{n} \left(\frac{q}{cX}\right)^r \left(\frac{(r-1)!\,r!}{(2r)!}\right) + \int \frac{dx}{X}\right] \end{cases}$$

4. $$\int \frac{x\,dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}$$

5. $$\int \frac{x\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q} \int \frac{dx}{X}$$

6. $$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}$$

7. $$\int \frac{x^2}{X}\,dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2-2ac}{2c^2} \int \frac{dx}{X}$$

8. $$\int \frac{x^2}{X^2}\,dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}$$

9. $$\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1}\,dx}{X^{n+1}} + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2}\,dx}{X^{n+1}}$$

20. $$\int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}$$

21. $$\int \frac{dx}{x^2X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right) \int \frac{dx}{X}$$

22. $$\int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}$$

INTEGRALS (Continued)

123. $$\int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}} - \frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}}$$

FORMS CONTAINING $\sqrt{a+bx}$

124. $$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}$$

125. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$$

126. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

127. $$\int x^m\sqrt{a+bx}\,dx = \begin{cases} \dfrac{2}{b(2m+3)}\left[x^m\sqrt{(a+bx)^3} - ma\displaystyle\int x^{m-1}\sqrt{a+bx}\,dx\right] \\ \text{or} \\ \dfrac{2}{b^{m+1}}\sqrt{a+bx}\displaystyle\sum_{r=0}^{m}\frac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)}(a+bx)^{r+1} \end{cases}$$

128. $$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}$$

129. $$\int \frac{\sqrt{a+bx}}{x^2}\,dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{a+bx}}$$

130. $$\int \frac{\sqrt{a+bx}}{x^m}\,dx = -\frac{1}{(m-1)a}\left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2}\int\frac{\sqrt{a+bx}}{x^{m-1}}\,dx\right]$$

131. $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$$

132. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}$$

133. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}$$

INTEGRALS (Continued)

134. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{(2m+1)b}\left[x^m\sqrt{a+bx} - ma\displaystyle\int \frac{x^{m-1}\,dx}{\sqrt{a+bx}}\right] \\ \text{or} \\ \dfrac{2(-a)^m\sqrt{a+bx}}{b^{m+1}} \displaystyle\sum_{r=0}^{m} \frac{(-1)^r m!(a+bx)^r}{(2r+1)r!(m-r)!a^r} \end{cases}$$

135. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right), \qquad (a > 0)$$

136. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}, \qquad (a < 0)$$

137. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}$$

138. $$\int \frac{dx}{x^n\sqrt{a+bx}} = \begin{cases} -\dfrac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \dfrac{(2n-3)b}{(2n-2)a}\displaystyle\int \frac{dx}{x^{n-1}\sqrt{a+bx}} \\ \text{or} \\ \dfrac{(2n-2)!}{[(n-1)!]^2}\left[-\dfrac{\sqrt{a+bx}}{a}\displaystyle\sum_{r=1}^{n-1} \frac{r!(r-1)!}{x^r(2r)!}\left(-\frac{b}{4a}\right)^{n-r-1} \right. \\ \qquad\qquad \left. + \left(-\dfrac{b}{4a}\right)^{n-1}\displaystyle\int \frac{dx}{x\sqrt{a+bx}}\right] \end{cases}$$

139. $$\int (a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}$$

140. $$\int x(a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right]$$

141. $$\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{\frac{m}{2}}}$$

142. $$\int \frac{(a+bx)^{\frac{n}{2}}\,dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}\,dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}\,dx$$

143. $$\int f(x, \sqrt{a+bx})\,dx = \frac{2}{b}\int f\left(\frac{z^2-a}{b}, z\right)z\,dz, \qquad (z = \sqrt{a+bx})$$

INTEGRALS (Continued)

FORMS CONTAINING $\sqrt{a+bx}$ and $\sqrt{c+dx}$

$$u = a + bx \qquad v = c + dx \qquad k = ad - bc$$

If $k = 0$, then $v = \frac{c}{a}u$, and formulas starting with 124 should be used in place of these.

144. $$\int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bduv}}{bv}, \ bd > o, \ k < o \\ \text{or} \\ \frac{2}{\sqrt{bd}} \tanh^{-1} \frac{\sqrt{bduv}}{du}, \ bd > o, \ k > o. \\ \text{or} \\ \frac{1}{\sqrt{bd}} \log \frac{(bv + \sqrt{bduv})^2}{v}, \qquad (bd > 0) \end{cases}$$

145. $$\int \frac{dx}{\sqrt{uv}} = \begin{cases} \frac{2}{\sqrt{-bd}} \tan^{-1} \frac{\sqrt{-bduv}}{bv}. \\ \text{or} \\ -\frac{1}{\sqrt{-bd}} \sin^{-1}\left(\frac{2bdx + ad + bc}{|k|}\right), \qquad (bd < 0) \end{cases}$$

146. $$\int \sqrt{uv}\, dx = \frac{k + 2bv}{4bd}\sqrt{uv} - \frac{k^2}{8bd}\int \frac{dx}{\sqrt{uv}}$$

147. $$\int \frac{dx}{v\sqrt{u}} = \begin{cases} \frac{1}{\sqrt{kd}} \log \frac{d\sqrt{u} - \sqrt{kd}}{d\sqrt{u} + \sqrt{kd}} \\ \text{or} \\ \frac{1}{\sqrt{kd}} \log \frac{(d\sqrt{u} - \sqrt{kd})^2}{v}, \qquad (kd > 0) \end{cases}$$

148. $$\int \frac{dx}{v\sqrt{u}} = \frac{2}{\sqrt{-kd}} \tan^{-1} \frac{d\sqrt{u}}{\sqrt{-kd}}, \qquad (kd < 0)$$

149. $$\int \frac{x\, dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad + bc}{2bd}\int \frac{dx}{\sqrt{uv}}$$

150. $$\int \frac{dx}{v\sqrt{uv}} = \frac{-2\sqrt{uv}}{kv}$$

INTEGRALS (Continued)

151. $\displaystyle\int \frac{v\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b}\int \frac{dx}{\sqrt{uv}}$

152. $\displaystyle\int \sqrt{\frac{v}{u}}\,dx = \frac{v}{|v|}\int \frac{v\,dx}{\sqrt{uv}}$

153. $\displaystyle\int v^m\sqrt{u}\,dx = \frac{1}{(2m+3)d}\left(2v^{m+1}\sqrt{u} + k\int \frac{v^m\,dx}{\sqrt{u}}\right)$

154. $\displaystyle\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left(\frac{\sqrt{u}}{v^{m-1}} + \left(m - \frac{3}{2}\right)b\int \frac{dx}{v^{m-1}\sqrt{u}}\right)$

155. $$\int \frac{v^m\,dx}{\sqrt{u}} = \begin{cases} \dfrac{2}{b(2m+1)}\left[v^m\sqrt{u} - mk\displaystyle\int \frac{v^{m-1}}{\sqrt{u}}\,dx\right] \\ \text{or} \\ \dfrac{2(m!)^2\sqrt{u}}{b(2m+1)!}\displaystyle\sum_{r=0}^{m}\left(-\frac{4k}{b}\right)^{m-r}\frac{(2r)!}{(r!)^2}v^r \end{cases}$$

FORMS CONTAINING $\sqrt{x^2 \pm a^2}$

156. $\displaystyle\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})]$

157. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$

158. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$

159. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$

160. $\displaystyle\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$

161. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - |a|\sec^{-1}\frac{x}{a}$

162. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

163. $\displaystyle\int x\sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 \pm a^2)^3}$

INTEGRALS (Continued)

164. $$\int \sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)\right]$$

165. $$\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

166. $$\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

167. $$\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}$$

168. $$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

169. $$\int x^3\sqrt{x^2 + a^2}\,dx = \left(\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2\right)\sqrt{(a^2 + x^2)^3}$$

170. $$\int x^3\sqrt{x^2 - a^2}\,dx = \frac{1}{5}\sqrt{(x^2 - a^2)^5} + \frac{a^2}{3}\sqrt{(x^2 - a^2)^3}$$

171. $$\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

172. $$\int \frac{x^3\,dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} \mp a^2\sqrt{x^2 \pm a^2}$$

173. $$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2x}$$

174. $$\int \frac{dx}{x^3\sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

175. $$\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2|a^3|}\sec^{-1}\frac{x}{a}$$

176. $$\int x^2\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{x}{6}\sqrt{(x^2 \pm a^2)^5} \mp \frac{a^2x}{24}\sqrt{(x^2 \pm a^2)^3} - \frac{a^4x}{16}\sqrt{x^2 \pm a^2} \mp \frac{a^6}{16}\log\left(x + \sqrt{x^2 \pm a^2}\right)$$

177. $$\int x^3\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{7}\sqrt{(x^2 \pm a^2)^7} \mp \frac{a^2}{5}\sqrt{(x^2 \pm a^2)^5}$$

INTEGRALS (Continued)

8. $$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log(x + \sqrt{x^2 \pm a^2})$$

9. $$\int \frac{\sqrt{x^2 + a^2}}{x^3}\,dx = -\frac{\sqrt{x^2 + a^2}}{2x^2} - \frac{1}{2a}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

0. $$\int \frac{\sqrt{x^2 - a^2}}{x^3}\,dx = -\frac{\sqrt{x^2 - a^2}}{2x^2} + \frac{1}{2|a|}\sec^{-1}\frac{x}{a}$$

31. $$\int \frac{\sqrt{x^2 \pm a^2}}{x^4}\,dx = \mp\frac{\sqrt{(x^2 \pm a^2)^3}}{3a^2x^3}$$

32. $$\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log(x + \sqrt{x^2 \pm a^2})$$

33. $$\int \frac{x^3\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \sqrt{x^2 \pm a^2} \pm \frac{a^2}{\sqrt{x^2 \pm a^2}}$$

84. $$\int \frac{dx}{x\sqrt{(x^2 + a^2)^3}} = \frac{1}{a^2\sqrt{x^2 + a^2}} - \frac{1}{a^3}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

85. $$\int \frac{dx}{x\sqrt{(x^2 - a^2)^3}} = -\frac{1}{a^2\sqrt{x^2 - a^2}} - \frac{1}{|a^3|}\sec^{-1}\frac{x}{a}$$

86. $$\int \frac{dx}{x^2\sqrt{(x^2 \pm a^2)^3}} = -\frac{1}{a^4}\left[\frac{\sqrt{x^2 \pm a^2}}{x} + \frac{x}{\sqrt{x^2 \pm a^2}}\right]$$

187. $$\int \frac{dx}{x^3\sqrt{(x^2 + a^2)^3}} = -\frac{1}{2a^2x^2\sqrt{x^2 + a^2}} - \frac{3}{2a^4\sqrt{x^2 + a^2}} + \frac{3}{2a^5}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

188. $$\int \frac{dx}{x^3\sqrt{(x^2 - a^2)^3}} = \frac{1}{2a^2x^2\sqrt{x^2 - a^2}} - \frac{3}{2a^4\sqrt{x^2 - a^2}} - \frac{3}{2|a^5|}\sec^{-1}\frac{x}{a}$$

189. $$\int \frac{x^m}{\sqrt{x^2 \pm a^2}}\,dx = \frac{1}{m}x^{m-1}\sqrt{x^2 \pm a^2} \mp \frac{m-1}{m}a^2\int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}}\,dx$$

190. $$\int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}}\,dx = \frac{(2m)!}{2^{2m}(m!)^2}\left[\sqrt{x^2 \pm a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{(2r)!}(\mp a^2)^{m-r}(2x)^{2r-1} + (\mp a^2)^m \log(x + \sqrt{x^2 \pm a^2})\right]$$

191. $$\int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}}\,dx = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(\mp 4a^2)^{m-r}x^{2r}$$

INTEGRALS (Continued)

192. $$\int \frac{dx}{x^m\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2x^{m-1}} \mp \frac{(m-2)}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{x^2 \pm a^2}}$$

193. $$\int \frac{dx}{x^{2m}\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r}x^{2r+1}}$$

194. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 + a^2}}{a^2}\sum_{r=1}^{m}(-1)^{m-r+1}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{(-1)^{m+1}}{2^{2m}a^{2m+1}}\log\frac{\sqrt{x^2+a^2}+}{x}\right.$$

195. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 - a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 - a^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}|a|^{2m+1}}\sec^{-1}\frac{x}{a}\right.$$

196. $$\int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}$$

197. $$\int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}$$

198. $$\int f(x, \sqrt{x^2 + a^2})\,dx = a\int f(a\tan u, a\sec u)\sec^2 u\,du, \qquad \left(u = \tan^{-1}\frac{x}{a}, a > 0\right)$$

199. $$\int f(x, \sqrt{x^2 - a^2})\,dx = a\int f(a\sec u, a\tan u)\sec u\tan u\,du, \qquad \left(u = \sec^{-1}\frac{x}{a}, a > 0\right)$$

FORMS CONTAINING $\sqrt{a^2 - x^2}$

200. $$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{|a|}\right]$$

201. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1}\dfrac{x}{|a|} \\ \text{or} \\ -\cos^{-1}\dfrac{x}{|a|} \end{cases}$$

202. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$$

INTEGRALS (Continued)

203. $\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a \log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$

204. $\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$

205. $\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}$

206. $\int \sqrt{(a^2 - x^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{|a|}\right]$

207. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$

208. $\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$

209. $\int x\sqrt{(a^2 - x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$

210. $\int x^2\sqrt{a^2 - x^2}\,dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{|a|}\right)$

211. $\int x^3\sqrt{a^2 - x^2}\,dx = (-\frac{1}{5}x^2 - \frac{2}{15}a^2)\sqrt{(a^2 - x^2)^3}$

212. $\int x^2\sqrt{(a^2 - x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2 - x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2 - x^2)^3} + \frac{a^4x}{16}\sqrt{a^2 - x^2} + \frac{a^6}{16}\sin^{-1}\frac{x}{|a|}$

213. $\int x^3\sqrt{(a^2 - x^2)^3}\,dx = \frac{1}{7}\sqrt{(a^2 - x^2)^7} - \frac{a^2}{5}\sqrt{(a^2 - x^2)^5}$

214. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{|a|}$

215. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}$

216. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{|a|}$

217. $\int \frac{\sqrt{a^2 - x^2}}{x^3}\,dx = -\frac{\sqrt{a^2 - x^2}}{2x^2} + \frac{1}{2a}\log\frac{a + \sqrt{a^2 - x^2}}{x}$

218. $\int \frac{\sqrt{a^2 - x^2}}{x^4}\,dx = -\frac{\sqrt{(a^2 - x^2)^3}}{3a^2x^3}$

INTEGRALS (Continued)

219. $$\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{|a|}$$

220. $$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = -\frac{2}{3}(a^2-x^2)^{\frac{3}{2}} - x^2(a^2-x^2)^{\frac{1}{2}} = -\frac{1}{3}\sqrt{a^2-x^2}(x^2+2a^2)$$

221. $$\int \frac{x^3\,dx}{\sqrt{(a^2-x^2)^3}} = 2(a^2-x^2)^{\frac{1}{2}} + \frac{x^2}{(a^2-x^2)^{\frac{1}{2}}} = -\frac{a^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}$$

222. $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

223. $$\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

224. $$\int \frac{dx}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left[-\frac{\sqrt{a^2-x^2}}{x} + \frac{x}{\sqrt{a^2-x^2}}\right]$$

225. $$\int \frac{dx}{x^3\sqrt{(a^2-x^2)^3}} = -\frac{1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

226. $$\int \frac{x^m}{\sqrt{a^2-x^2}}\,dx = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{\sqrt{a^2-x^2}}\,dx$$

227. $$\int \frac{x^{2m}}{\sqrt{a^2-x^2}}\,dx = \frac{(2m)!}{(m!)^2}\left[-\sqrt{a^2-x^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2^{2m-2r+1}(2r)!}a^{2m-2r}x^{2r-1} + \frac{a^{2m}}{2^{2m}}\sin^{-1}\frac{x}{|a|}\right]$$

228. $$\int \frac{x^{2m+1}}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(4a^2)^{m-r}x^{2r}$$

229. $$\int \frac{dx}{x^m\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(m-1)a^2x^{m-1}} + \frac{m-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{a^2-x^2}}$$

230. $$\int \frac{dx}{x^{2m}\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\sum_{r=0}^{m-1}\frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}$$

231. $$\int \frac{dx}{x^{2m+1}\sqrt{a^2-x^2}} = \frac{(2m)!}{(m!)^2}\left[-\frac{\sqrt{a^2-x^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}a^{2m+1}}\log\frac{a-\sqrt{a^2-x^2}}{x}\right]$$

INTEGRALS (Continued)

232. $$\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{2b\sqrt{a^2 - b^2}} \log \frac{(b\sqrt{a^2 - x^2} + x\sqrt{a^2 - b^2})^2}{b^2 - x^2}, \qquad (a^2 > b^2)$$

233. $$\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{b^2 - a^2}} \tan^{-1} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}, \qquad (b^2 > a^2)$$

234. $$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \tan^{-1} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}$$

235. $$\int \frac{\sqrt{a^2 - x^2}}{b^2 + x^2}\, dx = \frac{\sqrt{a^2 + b^2}}{|b|} \sin^{-1} \frac{x\sqrt{a^2 + b^2}}{|a|\sqrt{x^2 + b^2}} - \sin^{-1} \frac{x}{|a|}$$

236. $$\int f(x, \sqrt{a^2 - x^2})\, dx = a \int f(a \sin u, a \cos u) \cos u\, du, \qquad \left(u = \sin^{-1} \frac{x}{a},\ a > 0\right)$$

FORMS CONTAINING $\sqrt{a + bx + cx^2}$

$$X = a + bx + cx^2, q = 4ac - b^2, \text{and } k = \frac{4c}{q}$$

If $q = 0$, then $\sqrt{X} = \sqrt{c}\left|x + \frac{b}{2c}\right|$

237. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{c}} \log (2\sqrt{cX} + 2cx + b) \\ \quad \text{or} \\ \dfrac{1}{\sqrt{c}} \sinh^{-1} \dfrac{2cx + b}{\sqrt{q}}, \qquad (c > 0) \end{cases}$$

238. $$\int \frac{dx}{\sqrt{X}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}, \qquad (c < 0)$$

239. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$$

240. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}\left(\frac{1}{X} + 2k\right)$$

241. $$\int \frac{dx}{X^n\sqrt{X}} = \begin{cases} \dfrac{2(2cx + b)\sqrt{X}}{(2n - 1)qX^n} + \dfrac{2k(n - 1)}{2n - 1} \displaystyle\int \frac{dx}{X^{n-1}\sqrt{X}} \\ \quad \text{or} \\ \dfrac{(2cx + b)(n!)(n - 1)!4^n k^{n-1}}{q[(2n)!]\sqrt{X}} \displaystyle\sum_{r=0}^{n-1} \frac{(2r)!}{(4kX)^r (r!)^2} \end{cases}$$

INTEGRALS (Continued)

242. $$\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$$

243. $$\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$$

244. $$\int X^2\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$$

245. $$\int X^n\sqrt{X}\,dx = \begin{cases} \dfrac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \dfrac{2n+1}{2(n+1)k}\displaystyle\int X^{n-1}\sqrt{X}\,dx \\ \text{or} \\ \dfrac{(2n+2)!}{[(n+1)!]^2(4k)^{n+1}}\left[\dfrac{k(2cx+b)\sqrt{X}}{c}\displaystyle\sum_{r=0}^{n}\dfrac{r!(r+1)!(4kX)^r}{(2r+2)!} + \displaystyle\int \dfrac{dx}{\sqrt{X}}\right] \end{cases}$$

246. $$\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}$$

247. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$$

248. $$\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}$$

249. $$\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}$$

250. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}$$

251. $$\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}$$

252. $$\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}$$

253. $$\int \frac{x^n\,dx}{\sqrt{X}} = \frac{1}{nc}x^{n-1}\sqrt{X} - \frac{(2n-1)b}{2nc}\int \frac{x^{n-1}\,dx}{\sqrt{X}} - \frac{(n-1)a}{nc}\int \frac{x^{n-2}\,dx}{\sqrt{X}}$$

INTEGRALS (Continued)

54. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b(2cx+b)}{8c^2}\sqrt{X} - \frac{b}{4ck}\int \frac{dx}{\sqrt{X}}$$

55. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx$$

56. $$\int xX^n\sqrt{X}\,dx = \frac{X^{n+1}\sqrt{X}}{(2n+3)c} - \frac{b}{2c}\int X^n\sqrt{X}\,dx$$

57. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int \sqrt{X}\,dx$$

58. $$\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\frac{2\sqrt{aX}+bx+2a}{x}, \qquad (a>0)$$

59. $$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx+2a}{|x|\sqrt{-q}}\right), \qquad (a<0)$$

60. $$\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \qquad (a=0)$$

261. $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{X}}$$

262. $$\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}$$

263. $$\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}$$

FORMS INVOLVING $\sqrt{2ax - x^2}$

264. $$\int \sqrt{2ax-x^2}\,dx = \frac{1}{2}\left[(x-a)\sqrt{2ax-x^2} + a^2\sin^{-1}\frac{x-a}{|a|}\right]$$

265. $$\int \frac{dx}{\sqrt{2ax-x^2}} = \begin{cases} \cos^{-1}\dfrac{a-x}{|a|} \\ \text{or} \\ \sin^{-1}\dfrac{x-a}{|a|} \end{cases}$$

INTEGRALS (Continued)

266. $$\int x^n\sqrt{2ax - x^2}\,dx = \begin{cases} -\dfrac{x^{n-1}(2ax - x^2)^{3/2}}{n+2} + \dfrac{(2n+1)a}{n+2}\displaystyle\int x^{n-1}\sqrt{2ax - x^2}\,dx \\ \text{or} \\ \sqrt{2ax - x^2}\left[\dfrac{x^{n+1}}{n+2} - \displaystyle\sum_{r=0}^{n}\dfrac{(2n+1)!(r!)^2 a^{n-r+1}}{2^{n-r}(2r+1)!(n+2)!n!}x^r\right] \\ \qquad + \dfrac{(2n+1)!a^{n+2}}{2^n n!(n+2)!}\sin^{-1}\dfrac{x -}{|a|} \end{cases}$$

267. $$\int\frac{\sqrt{2ax - x^2}}{x^n}\,dx = \frac{(2ax - x^2)^{3/2}}{(3 - 2n)ax^n} + \frac{n-3}{(2n-3)a}\int\frac{\sqrt{2ax - x^2}}{x^{n-1}}\,dx$$

268. $$\int\frac{x^n\,dx}{\sqrt{2ax - x^2}} = \begin{cases} \dfrac{-x^{n-1}\sqrt{2ax - x^2}}{n} + \dfrac{a(2n-1)}{n}\displaystyle\int\dfrac{x^{n-1}}{\sqrt{2ax - x^2}}\,dx \\ \text{or} \\ -\sqrt{2ax - x^2}\displaystyle\sum_{r=1}^{n}\dfrac{(2n)!r!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2}x^{r-1} \\ \qquad + \dfrac{(2n)!a^n}{2^n(n!)^2}\sin^{-1}\dfrac{x - a}{|a|} \end{cases}$$

269. $$\int\frac{dx}{x^n\sqrt{2ax - x^2}} = \begin{cases} \dfrac{\sqrt{2ax - x^2}}{a(1 - 2n)x^n} + \dfrac{n-1}{(2n-1)a}\displaystyle\int\dfrac{dx}{x^{n-1}\sqrt{2ax - x^2}} \\ \text{or} \\ -\sqrt{2ax - x^2}\displaystyle\sum_{r=0}^{n-1}\dfrac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2 a^{n-r}x^{r+1}} \end{cases}$$

270. $$\int\frac{dx}{(2ax - x^2)^{3/2}} = \frac{x - a}{a^2\sqrt{2ax - x^2}}$$

271. $$\int\frac{x\,dx}{(2ax - x^2)^{3/2}} = \frac{x}{a\sqrt{2ax - x^2}}$$

MISCELLANEOUS ALGEBRAIC FORMS

272. $$\int\frac{dx}{\sqrt{2ax + x^2}} = \log\left(x + a + \sqrt{2ax + x^2}\right)$$

273. $$\int\sqrt{ax^2 + c}\,dx = \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}}\log\left(x\sqrt{a} + \sqrt{ax^2 + c}\right), \qquad (a > 0)$$

274. $$\int\sqrt{ax^2 + c}\,dx = \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \qquad (a < 0)$$

INTEGRALS (Continued)

275. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1} x - \sqrt{1-x^2}$$

276. $$\int \frac{dx}{x\sqrt{ax^n + c}} = \begin{cases} \dfrac{1}{n\sqrt{c}} \log \dfrac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{ax^n + c} + \sqrt{c}} \\ \text{or} \\ \dfrac{2}{n\sqrt{c}} \log \dfrac{\sqrt{ax^n + c} - \sqrt{c}}{\sqrt{x^n}}, \qquad (c > 0) \end{cases}$$

277. $$\int \frac{dx}{x\sqrt{ax^n + c}} = \frac{2}{n\sqrt{-c}} \sec^{-1} \sqrt{-\frac{ax^n}{c}}, \qquad (c < 0)$$

278. $$\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{a}} \log \left(x\sqrt{a} + \sqrt{ax^2 + c}\right), \qquad (a > 0)$$

279. $$\int \frac{dx}{\sqrt{ax^2 + c}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(x\sqrt{-\frac{a}{c}}\right), \qquad (a < 0)$$

280. $$\int (ax^2 + c)^{m+\frac{1}{2}}\,dx = \begin{cases} \dfrac{x(ax^2 + c)^{m+\frac{1}{2}}}{2(m+1)} + \dfrac{(2m+1)c}{2(m+1)} \displaystyle\int (ax^2 + c)^{m-\frac{1}{2}}\,dx \\ \text{or} \\ x\sqrt{ax^2 + c} \displaystyle\sum_{r=0}^{m} \dfrac{(2m+1)!(r!)^2 c^{m-r}}{2^{2m-2r+1} m!(m+1)!(2r+1)!} (ax^2 + c)^r \\ \quad + \dfrac{(2m+1)!c^{m+1}}{2^{2m+1} m!(m+1)!} \displaystyle\int \dfrac{dx}{\sqrt{ax^2 + c}} \end{cases}$$

281. $$\int x(ax^2 + c)^{m+\frac{1}{2}}\,dx = \frac{(ax^2 + c)^{m+\frac{3}{2}}}{(2m+3)a}$$

282. $$\int \frac{(ax^2 + c)^{m+\frac{1}{2}}}{x}\,dx = \begin{cases} \dfrac{(ax^2 + c)^{m+\frac{1}{2}}}{2m+1} + c \displaystyle\int \dfrac{(ax^2 + c)^{m-\frac{1}{2}}}{x}\,dx \\ \text{or} \\ \sqrt{ax^2 + c} \displaystyle\sum_{r=0}^{m} \dfrac{c^{m-r}(ax^2 + c)^r}{2r+1} + c^{m+1} \displaystyle\int \dfrac{dx}{x\sqrt{ax^2 + c}} \end{cases}$$

283. $$\int \frac{dx}{(ax^2 + c)^{m+\frac{1}{2}}} = \begin{cases} \dfrac{x}{(2m-1)c(ax^2 + c)^{m-\frac{1}{2}}} + \dfrac{2m-2}{(2m-1)c} \displaystyle\int \dfrac{dx}{(ax^2 + c)^{m-\frac{1}{2}}} \\ \text{or} \\ \dfrac{x}{\sqrt{ax^2 + c}} \displaystyle\sum_{r=0}^{m-1} \dfrac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2 c^{m-r}(ax^2 + c)^r} \end{cases}$$

284. $\int \frac{dx}{x^m\sqrt{ax^2+c}} = -\frac{\sqrt{ax^2+c}}{(m-1)cx^{m-1}} - \frac{(}{(m}$

285. $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\log\frac{x\sqrt{2}+\sqrt{}}{1-x^2}$

286. $\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1+x^4}}$

287. $\int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\log\frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}$

288. $\int \frac{dx}{x\sqrt{x^n-a^2}} = -\frac{2}{na}\sin^{-1}\frac{a}{\sqrt{x^n}}$

289. $\int \sqrt{\frac{x}{a^3-x^3}}\,dx = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}$

FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

290. $\int (\sin ax)\,dx = -\frac{1}{a}\cos ax$

291. $\int (\cos ax)\,dx = \frac{1}{a}\sin ax$

292. $\int (\tan ax)\,dx = -\frac{1}{a}\log\cos ax = \frac{1}{a}\log\sec ax$

293. $\int (\cot ax)\,dx = \frac{1}{a}\log\sin ax = -\frac{1}{a}\log\csc ax$

294. $\int (\sec ax)\,dx = \frac{1}{a}\log(\sec ax + \tan ax) = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$

295. $\int (\csc ax)\,dx = \frac{1}{a}\log(\csc ax - \cot ax) = \frac{1}{a}\log\tan\frac{ax}{2}$

?96. $\int (\sin^2 ax)\,dx = -\frac{1}{2a}\cos ax\sin ax + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$

?7. $\int (\sin^3 ax)\,dx = -\frac{1}{3a}(\cos ax)(\sin^2 ax + 2)$

$\int (\sin^4 ax)\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

$\sin^n ax)\,dx = -\frac{\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int($

...LS (Continued)

$\ldots \frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

$\ldots \frac{\ldots x}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\sin^{2r} ax$

$\ldots ax \cos ax + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$

$\ldots \frac{\ldots}{3a}(\sin ax)(\cos^2 ax + 2)$

$\ldots = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

$\ldots x)\,dx = \frac{1}{na}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int(\cos^{n-2} ax)\,dx$

$\ldots os^{2m} ax)\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

$\ldots \int(\cos^{2m+1} ax)\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\cos^{2r} ax$

308. $\int\frac{dx}{\sin^2 ax} = \int(\csc^2 ax)\,dx = -\frac{1}{a}\cot ax$

309. $\int\frac{dx}{\sin^m ax} = \int(\csc^m ax)\,dx = -\frac{1}{(m-1)a}\cdot\frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1}\int\frac{dx}{\sin^{m-2} ax}$

310. $\int\frac{dx}{\sin^{2m} ax} = \int(\csc^{2m} ax)\,dx = -\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\sin^{2r+1} ax}$

311. $\int\frac{dx}{\sin^{2m+1} ax} = \int(\csc^{2m+1} ax)\,dx =$

$-\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)!\sin^{2r+2} ax} + \frac{1}{a}\cdot\frac{(2m)!}{2^{2m}(m!)^2}\log\tan\frac{ax}{\ldots}$

312. $\int\frac{dx}{\cos^2 ax} = \int(\sec^2 ax)\,dx = \frac{1}{a}\tan ax$

$\ldots = \int(\sec^n ax)\,dx = \frac{1}{(n-1)a}\cdot\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2} ax}$

$\ldots m\ ax)\,dx = \frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\cos^{2r+1} ax}$

$\ldots dx \ldots$

INTEGRALS (Continued)

315. $$\int \frac{dx}{\cos^{2m+1} ax} = \int (\sec^{2m+1} ax)\, dx =$$
$$\frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \cos^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log(\sec ax + \tan ax)$$

316. $$\int (\sin mx)(\sin nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

317. $$\int (\cos mx)(\cos nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

318. $$\int (\sin ax)(\cos ax)\, dx = \frac{1}{2a} \sin^2 ax$$

319. $$\int (\sin mx)(\cos nx)\, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

320. $$\int (\sin^2 ax)(\cos^2 ax)\, dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}$$

321. $$\int (\sin ax)(\cos^m ax)\, dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

322. $$\int (\sin^m ax)(\cos ax)\, dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

323. $$\int (\cos^m ax)(\sin^n ax)\, dx = \begin{cases} \dfrac{\cos^{m-1} ax \sin^{n+1} ax}{(m+n)a} + \dfrac{m-1}{m+n} \displaystyle\int (\cos^{m-2} ax)(\sin^n ax)\, dx \\ \text{or} \\ -\dfrac{\sin^{n-1} ax \cos^{m+1} ax}{(m+n)a} + \dfrac{n-1}{m+n} \displaystyle\int (\cos^m ax)(\sin^{n-2} ax)\, dx \end{cases}$$

324. $$\int \frac{\cos^m ax}{\sin^n ax} dx = \begin{cases} -\dfrac{\cos^{m+1} ax}{(n-1)a \sin^{n-1} ax} - \dfrac{m-n+2}{n-1} \displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax} dx \\ \text{or} \\ \dfrac{\cos^{m-1} ax}{a(m-n) \sin^{n-1} ax} + \dfrac{m-1}{m-n} \displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax} dx \end{cases}$$

INTEGRALS (Continued)

25. $$\int \frac{\sin^m ax}{\cos^n ax}\,dx = \begin{cases} \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} - \dfrac{m-n+2}{n-1}\displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax}\,dx \\ \text{or} \\ -\dfrac{\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} + \dfrac{m-1}{m-n}\displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax}\,dx \end{cases}$$

26. $$\int \frac{\sin ax}{\cos^2 ax}\,dx = \frac{1}{a\cos ax} = \frac{\sec ax}{a}$$

27. $$\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{1}{a}\sin ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

28. $$\int \frac{\cos ax}{\sin^2 ax}\,dx = -\frac{1}{a\sin ax} = -\frac{\csc ax}{a}$$

29. $$\int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a}\log\tan ax$$

30. $$\int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a}\left(\sec ax + \log\tan\frac{ax}{2}\right)$$

31. $$\int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}$$

32. $$\int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a}\csc ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

33. $$\int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a}\cot 2ax$$

34. $$\int \frac{dx}{\sin^m ax\cos^n ax} = \begin{cases} -\dfrac{1}{a(m-1)(\sin^{m-1} ax)(\cos^{n-1} ax)} \\ \qquad + \dfrac{m+n-2}{m-1}\displaystyle\int \frac{dx}{(\sin^{m-2} ax)(\cos^n ax)} \\ \text{or} \\ \dfrac{1}{a(n-1)\sin^{m-1} ax\cos^{n-1} ax} \\ \qquad - \dfrac{m+n-2}{n-1}\displaystyle\int \frac{dx}{\sin^m ax\cos^{n-2} ax} \end{cases}$$

35. $$\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx)$$

36. $$\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx)$$

37. $$\int \frac{dx}{1 \pm \sin ax} = \mp\frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

INTEGRALS (Continued)

338. $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

339. $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

*340. $$\int \frac{dx}{a+b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\dfrac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} \\ \text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}}\log\dfrac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \end{cases}$$

*341. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\dfrac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b} \\ \text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}}\log\left(\dfrac{\sqrt{b^2-a^2}\tan\frac{x}{2}+a+b}{\sqrt{b^2-a^2}\tan\frac{x}{2}-a-b}\right) \end{cases}$$

*342. $$\int \frac{dx}{a+b\sin x+c\cos x}$$

$$= \begin{cases} \dfrac{1}{\sqrt{b^2+c^2-a^2}}\log\dfrac{b-\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}{b+\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}, & \text{if } a^2 < b^2+c^2, a \neq c \\ \text{or} \\ \dfrac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\dfrac{b+(a-c)\tan\frac{x}{2}}{\sqrt{a^2-b^2-c^2}}, & \text{if } a^2 > b^2+c^2 \\ \text{or} \\ \dfrac{1}{a}\left[\dfrac{a-(b+c)\cos x-(b-c)\sin x}{a-(b-c)\cos x+(b+c)\sin x}\right], & \text{if } a^2 = b^2+c^2, a \neq c. \end{cases}$$

*343. $$\int \frac{\sin^2 x\,dx}{a+b\cos^2 x} = \frac{1}{b}\sqrt{\frac{a+b}{a}}\tan^{-1}\left(\sqrt{\frac{a}{a+b}}\tan x\right) - \frac{x}{b}, \qquad (ab > 0, \text{ or } |a| > |b|)$$

*See note 6–page 336.

INTEGRALS (Continued)

4. $$\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1} \left(\frac{b \tan x}{a} \right)$$

5. $$\int \frac{\cos^2 cx}{a^2 + b^2 \sin^2 cx} dx = \frac{\sqrt{a^2 + b^2}}{ab^2 c} \tan^{-1} \frac{\sqrt{a^2 + b^2} \tan cx}{a} - \frac{x}{b^2}$$

6. $$\int \frac{\sin cx \cos cx}{a \cos^2 cx + b \sin^2 cx} dx = \frac{1}{2c(b - a)} \log (a \cos^2 cx + b \sin^2 cx)$$

7. $$\int \frac{\cos cx}{a \cos cx + b \sin cx} dx = \int \frac{dx}{a + b \tan cx} =$$
$$\frac{1}{c(a^2 + b^2)} [acx + b \log (a \cos cx + b \text{ si}$$

48. $$\int \frac{\sin cx}{a \sin cx + b \cos cx} dx = \int \frac{dx}{a + b \cot cx} =$$
$$\frac{1}{c(a^2 + b^2)} [acx - b \log (a \sin cx + b \text{ cc}$$

49. $$\int \frac{dx}{a \cos^2 x + 2b \cos x \sin x + c \sin^2 x} = \begin{cases} \dfrac{1}{2\sqrt{b^2 - ac}} \log \dfrac{c \tan x + b - \sqrt{b^2}}{c \tan x + b + \sqrt{b^2}} \\ (b^2 \\ \text{or} \\ \dfrac{1}{\sqrt{ac - b^2}} \tan^{-1} \dfrac{c \tan x + b}{\sqrt{ac - b^2}}, \quad (b^2 \\ \text{or} \\ -\dfrac{1}{c \tan x + b}, \quad (b^2 = ac) \end{cases}$$

350. $$\int \frac{\sin ax}{1 \pm \sin ax} dx = \pm x + \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right)$$

351. $$\int \frac{dx}{(\sin ax)(1 \pm \sin ax)} = \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right) + \frac{1}{a} \log \tan \frac{ax}{2}$$

352. $$\int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) - \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

353. $$\int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

354. $$\int \frac{\sin ax}{(1 + \sin ax)^2} dx = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$$

*See note 6–page 336.

INTEGRALS (Continued)

355. $$\int \frac{\sin ax}{(1-\sin ax)^2}\,dx = -\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

356. $$\int \frac{\sin x\,dx}{a+b\sin x} = \frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+b\sin x}$$

357. $$\int \frac{dx}{(\sin x)(a+b\sin x)} = \frac{1}{a}\log\tan\frac{x}{2}-\frac{b}{a}\int\frac{dx}{a+b\sin x}$$

358. $$\int \frac{dx}{(a+b\sin x)^2} = \frac{b\cos x}{(a^2-b^2)(a+b\sin x)}+\frac{a}{a^2-b^2}\int\frac{dx}{a+b\sin x}$$

359. $$\int \frac{\sin x\,dx}{(a+b\sin x)^2} = \frac{a\cos x}{(b^2-a^2)(a+b\sin x)}+\frac{b}{b^2-a^2}\int\frac{dx}{a+b\sin x}$$

*360. $$\int \frac{dx}{a^2+b^2\sin^2 cx} = \frac{1}{ac\sqrt{a^2+b^2}}\tan^{-1}\frac{\sqrt{a^2+b^2}\tan cx}{a}$$

*361. $$\int \frac{dx}{a^2-b^2\sin^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2-b^2}}\tan^{-1}\dfrac{\sqrt{a^2-b^2}\tan cx}{a}, & (a^2>b^2) \\ \text{or} & \\ \dfrac{1}{2ac\sqrt{b^2-a^2}}\log\dfrac{\sqrt{b^2-a^2}\tan cx+a}{\sqrt{b^2-a^2}\tan cx-a}, & (a^2<b^2) \end{cases}$$

362. $$\int \frac{\cos ax}{1+\cos ax}\,dx = x-\frac{1}{a}\tan\frac{ax}{2}$$

363. $$\int \frac{\cos ax}{1-\cos ax}\,dx = -x-\frac{1}{a}\cot\frac{ax}{2}$$

364. $$\int \frac{dx}{(\cos ax)(1+\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)-\frac{1}{a}\tan\frac{ax}{2}$$

365. $$\int \frac{dx}{(\cos ax)(1-\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)-\frac{1}{a}\cot\frac{ax}{2}$$

366. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2}+\frac{1}{6a}\tan^3\frac{ax}{2}$$

367. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2}-\frac{1}{6a}\cot^3\frac{ax}{2}$$

368. $$\int \frac{\cos ax}{(1+\cos ax)^2}\,dx = \frac{1}{2a}\tan\frac{ax}{2}-\frac{1}{6a}\tan^3\frac{ax}{2}$$

369. $$\int \frac{\cos ax}{(1-\cos ax)^2}\,dx = \frac{1}{2a}\cot\frac{ax}{2}-\frac{1}{6a}\cot^3\frac{ax}{2}$$

*See note 6–page 336.

INTEGRALS (Continued)

370. $$\int \frac{\cos x\,dx}{a + b\cos x} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a + b\cos x}$$

371. $$\int \frac{dx}{(\cos x)(a + b\cos x)} = \frac{1}{a}\log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - \frac{b}{a}\int \frac{dx}{a + b\cos x}$$

372. $$\int \frac{dx}{(a + b\cos x)^2} = \frac{b\sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2}\int \frac{dx}{a + b\cos x}$$

373. $$\int \frac{\cos x}{(a + b\cos x)^2}\,dx = \frac{a\sin x}{(a^2 - b^2)(a + b\cos x)} - \frac{b}{a^2 - b^2}\int \frac{dx}{a + b\cos x}$$

374. $$\int \frac{dx}{a^2 + b^2 - 2ab\cos cx} = \frac{2}{c(a^2 - b^2)}\tan^{-1}\left(\frac{a + b}{a - b}\tan\frac{cx}{2}\right)$$

375. $$\int \frac{dx}{a^2 + b^2\cos^2 cx} = \frac{1}{ac\sqrt{a^2 + b^2}}\tan^{-1}\frac{a\tan cx}{\sqrt{a^2 + b^2}}$$

376. $$\int \frac{dx}{a^2 - b^2\cos^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2 - b^2}}\tan^{-1}\dfrac{a\tan cx}{\sqrt{a^2 - b^2}}, & (a^2 > b^2) \\ \text{or} & \\ \dfrac{1}{2ac\sqrt{b^2 - a^2}}\log\dfrac{a\tan cx - \sqrt{b^2 - a^2}}{a\tan cx + \sqrt{b^2 - a^2}}, & (b^2 > a^2) \end{cases}$$

377. $$\int \frac{\sin ax}{1 \pm \cos ax}\,dx = \mp\frac{1}{a}\log(1 \pm \cos ax)$$

378. $$\int \frac{\cos ax}{1 \pm \sin ax}\,dx = \pm\frac{1}{a}\log(1 \pm \sin ax)$$

379. $$\int \frac{dx}{(\sin ax)(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\log\tan\frac{ax}{2}$$

380. $$\int \frac{dx}{(\cos ax)(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

381. $$\int \frac{\sin ax}{(\cos ax)(1 \pm \cos ax)}\,dx = \frac{1}{a}\log(\sec ax \pm 1)$$

382. $$\int \frac{\cos ax}{(\sin ax)(1 \pm \sin ax)}\,dx = -\frac{1}{a}\log(\csc ax \pm 1)$$

383. $$\int \frac{\sin ax}{(\cos ax)(1 \pm \sin ax)}\,dx = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

384. $$\int \frac{\cos ax}{(\sin ax)(1 \pm \cos ax)}\,dx = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a}\log\tan\frac{ax}{2}$$

*See note 6–page 336.

INTEGRALS (Continued)

385. $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log \tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

386. $$\int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a} \tan\left(ax \mp \frac{\pi}{4}\right)$$

387. $$\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log\left(1 \pm \tan \frac{ax}{2}\right)$$

388. $$\int \frac{dx}{a^2 \cos^2 cx - b^2 \sin^2 cx} = \frac{1}{2abc} \log \frac{b \tan cx + a}{b \tan cx - a}$$

389. $$\int x(\sin ax)\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$$

390. $$\int x^2(\sin ax)\, dx = \frac{2x}{a^2} \sin ax - \frac{a^2x^2 - 2}{a^3} \cos ax$$

391. $$\int x^3(\sin ax)\, dx = \frac{3a^2x^2 - 6}{a^4} \sin ax - \frac{a^2x^3 - 6x}{a^3} \cos ax$$

392. $$\int x^m \sin ax\, dx = \begin{cases} -\frac{1}{a} x^m \cos ax + \frac{m}{a} \int x^{m-1} \cos ax\, dx \\ \text{or} \\ \cos ax \sum_{r=0}^{\left[\frac{m}{2}\right]} (-1)^{r+1} \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ \quad + \sin ax \sum_{r=0}^{\left[\frac{m-1}{2}\right]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$; $[3\frac{1}{2}] = 3$, $[\frac{1}{2}] = 0$, etc.

393. $$\int x(\cos ax)\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$$

394. $$\int x^2(\cos ax)\, dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$$

395. $$\int x^3(\cos ax)\, dx = \frac{3a^2x^2 - 6}{a^4} \cos ax + \frac{a^2x^3 - 6x}{a^3} \sin ax$$

396. $$\int x^m(\cos ax)\, dx = \begin{cases} \frac{x^m \sin ax}{a} - \frac{m}{a} \int x^{m-1} \sin ax\, dx \\ \text{or} \\ \sin ax \sum_{r=0}^{\left[\frac{m}{2}\right]} (-1)^r \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ \quad + \cos ax \sum_{r=0}^{\left[\frac{m-1}{2}\right]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

See note integral 392.

INTEGRALS (Continued)

397. $$\int \frac{\sin ax}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}$$

398. $$\int \frac{\cos ax}{x}\,dx = \log x + \sum_{n=1}^{\infty} (-1)^n \frac{(ax)^{2n}}{2n(2n)!}$$

399. $$\int x(\sin^2 ax)\,dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

400. $$\int x^2(\sin^2 ax)\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax - \frac{x \cos 2ax}{4a^2}$$

401. $$\int x(\sin^3 ax)\,dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}$$

402. $$\int x(\cos^2 ax)\,dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$$

403. $$\int x^2(\cos^2 ax)\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax + \frac{x \cos 2ax}{4a^2}$$

404. $$\int x(\cos^3 ax)\,dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}$$

405. $$\int \frac{\sin ax}{x^m}\,dx = -\frac{\sin ax}{(m-1)x^{m-1}} + \frac{a}{m-1} \int \frac{\cos ax}{x^{m-1}}\,dx$$

406. $$\int \frac{\cos ax}{x^m}\,dx = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1} \int \frac{\sin ax}{x^{m-1}}\,dx$$

407. $$\int \frac{x}{1 \pm \sin ax}\,dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2} \log(1 \pm \sin ax)$$

408. $$\int \frac{x}{1 + \cos ax}\,dx = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log \cos \frac{ax}{2}$$

409. $$\int \frac{x}{1 - \cos ax}\,dx = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log \sin \frac{ax}{2}$$

410. $$\int \frac{x + \sin x}{1 + \cos x}\,dx = x \tan \frac{x}{2}$$

411. $$\int \frac{x - \sin x}{1 - \cos x}\,dx = -x \cot \frac{x}{2}$$

412. $$\int \sqrt{1 - \cos ax}\,dx = -\frac{2 \sin ax}{a\sqrt{1 - \cos ax}} = -\frac{2\sqrt{2}}{a} \cos\left(\frac{ax}{2}\right)$$

413. $$\int \sqrt{1 + \cos ax}\,dx = \frac{2 \sin ax}{a\sqrt{1 + \cos ax}} = \frac{2\sqrt{2}}{a} \sin\left(\frac{ax}{2}\right)$$

INTEGRALS (Continued)

414. $\int \sqrt{1 + \sin x}\, dx = \pm 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right),$

[use + if $(8k - 1)\frac{\pi}{2} < x \leq (8k + 3)\frac{\pi}{2}$, otherwise −; k an integer]

415. $\int \sqrt{1 - \sin x}\, dx = \pm 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right),$

[use + if $(8k - 3)\frac{\pi}{2} < x \leq (8k + 1)\frac{\pi}{2}$, otherwise −; k an integer]

416. $\int \frac{dx}{\sqrt{1 - \cos x}} = \pm \sqrt{2} \log \tan\frac{x}{4},$

[use + if $4k\pi < x < (4k + 2)\pi$, otherwise −; k an integer]

417. $\int \frac{dx}{\sqrt{1 + \cos x}} = \pm \sqrt{2} \log \tan\left(\frac{x + \pi}{4}\right),$

[use + if $(4k - 1)\pi < x < (4k + 1)\pi$, otherwise −; k an integer]

418. $\int \frac{dx}{\sqrt{1 - \sin x}} = \pm \sqrt{2} \log \tan\left(\frac{x}{4} - \frac{\pi}{8}\right),$

[use + if $(8k + 1)\frac{\pi}{2} < x < (8k + 5)\frac{\pi}{2}$, otherwise −; k an integer]

419. $\int \frac{dx}{\sqrt{1 + \sin x}} = \pm \sqrt{2} \log \tan\left(\frac{x}{4} + \frac{\pi}{8}\right),$

[use + if $(8k - 1)\frac{\pi}{2} < x < (8k + 3)\frac{\pi}{2}$, otherwise −; k an integer]

420. $\int (\tan^2 ax)\, dx = \frac{1}{a} \tan ax - x$

421. $\int (\tan^3 ax)\, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log \cos ax$

422. $\int (\tan^4 ax)\, dx = \frac{\tan^3 ax}{3a} - \frac{1}{a} \tan x + x$

423. $\int (\tan^n ax)\, dx = \frac{\tan^{n-1} ax}{a(n - 1)} - \int (\tan^{n-2} ax)\, dx$

424. $\int (\cot^2 ax)\, dx = -\frac{1}{a} \cot ax - x$

425. $\int (\cot^3 ax)\, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log \sin ax$

426. $\int (\cot^4 ax)\, dx = -\frac{1}{3a} \cot^3 ax + \frac{1}{a} \cot ax + x$

INTEGRALS (Continued)

427. $$\int (\cot^n ax)\,dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int (\cot^{n-2} ax)\,dx$$

428. $$\int \frac{x}{\sin^2 ax}\,dx = \int x(\csc^2 ax)\,dx = -\frac{x \cot ax}{a} + \frac{1}{a^2}\log \sin ax$$

429. $$\int \frac{x}{\sin^n ax}\,dx = \int x(\csc^n ax)\,dx = -\frac{x \cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{(n-2)}{(n-1)}\int \frac{x}{\sin^{n-2} ax}\,dx$$

430. $$\int \frac{x}{\cos^2 ax}\,dx = \int x(\sec^2 ax)\,dx = \frac{1}{a} x \tan ax + \frac{1}{a^2}\log \cos ax$$

431. $$\int \frac{x}{\cos^n ax}\,dx = \int x(\sec^n ax)\,dx = \frac{x \sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x}{\cos^{n-2} ax}\,dx$$

432. $$\int \frac{\sin ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}$$

433. $$\int \frac{\sin ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right)$$

434. $$\int (\sin ax)\sqrt{1+b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1+b^2\sin^2 ax} - \frac{1+b^2}{2ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}$$

435. $$\int (\sin ax)\sqrt{1-b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1-b^2\sin^2 ax} - \frac{1-b^2}{2ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right)$$

436. $$\int \frac{\cos ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = \frac{1}{ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

437. $$\int \frac{\cos ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = \frac{1}{ab}\sin^{-1}(b\sin ax)$$

438. $$\int (\cos ax)\sqrt{1+b^2\sin^2 ax}\,dx = \frac{\sin ax}{2a}\sqrt{1+b^2\sin^2 ax} + \frac{1}{2ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

INTEGRALS (Continued)

439. $$\int (\cos ax)\sqrt{1 - b^2 \sin^2 ax}\, dx = \frac{\sin ax}{2a}\sqrt{1 - b^2 \sin^2 ax} + \frac{1}{2ab}\sin^{-1}(b \sin ax)$$

440. $$\int \frac{dx}{\sqrt{a + b\tan^2 cx}} = \frac{\pm 1}{c\sqrt{a - b}}\sin^{-1}\left(\sqrt{\frac{a - b}{a}}\sin cx\right), \qquad (a > |b|)$$

[use + if $(2k - 1)\frac{\pi}{2} < x \leq (2k + 1)\frac{\pi}{2}$, otherwise $-$; k an integer]

FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

441. $$\int (\sin^{-1} ax)\, dx = x \sin^{-1} ax + \frac{\sqrt{1 - a^2x^2}}{a}$$

442. $$\int (\cos^{-1} ax)\, dx = x \cos^{-1} ax - \frac{\sqrt{1 - a^2x^2}}{a}$$

443. $$\int (\tan^{-1} ax)\, dx = x \tan^{-1} ax - \frac{1}{2a}\log(1 + a^2x^2)$$

444. $$\int (\cot^{-1} ax)\, dx = x \cot^{-1} ax + \frac{1}{2a}\log(1 + a^2x^2)$$

445. $$\int (\sec^{-1} ax)\, dx = x \sec^{-1} ax - \frac{1}{a}\log(ax + \sqrt{a^2x^2 - 1})$$

446. $$\int (\csc^{-1} ax)\, dx = x \csc^{-1} ax + \frac{1}{a}\log(ax + \sqrt{a^2x^2 - 1})$$

447. $$\int \left(\sin^{-1}\frac{x}{a}\right) dx = x \sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}, \qquad (a > 0)$$

448. $$\int \left(\cos^{-1}\frac{x}{a}\right) dx = x \cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}, \qquad (a > 0)$$

449. $$\int \left(\tan^{-1}\frac{x}{a}\right) dx = x \tan^{-1}\frac{x}{a} - \frac{a}{2}\log(a^2 + x^2)$$

450. $$\int \left(\cot^{-1}\frac{x}{a}\right) dx = x \cot^{-1}\frac{x}{a} + \frac{a}{2}\log(a^2 + x^2)$$

451. $$\int x[\sin^{-1}(ax)]\, dx = \frac{1}{4a^2}[(2a^2x^2 - 1)\sin^{-1}(ax) + ax\sqrt{1 - a^2x^2}]$$

452. $$\int x[\cos^{-1}(ax)]\, dx = \frac{1}{4a^2}[(2a^2x^2 - 1)\cos^{-1}(ax) - ax\sqrt{1 - a^2x^2}]$$

INTEGRALS (Continued)

453. $\int x^n[\sin^{-1}(ax)]\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}(ax) - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad (n \neq -1)$

454. $\int x^n[\cos^{-1}(ax)]\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}(ax) + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad (n \neq -1)$

455. $\int x(\tan^{-1} ax)\,dx = \frac{1+a^2x^2}{2a^2}\tan^{-1} ax - \frac{x}{2a}$

456. $\int x^n(\tan^{-1} ax)\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}}{1+a^2x^2}\,dx$

457. $\int x(\cot^{-1} ax)\,dx = \frac{1+a^2x^2}{2a^2}\cot^{-1} ax + \frac{x}{2a}$

458. $\int x^n(\cot^{-1} ax)\,dx = \frac{x^{n+1}}{n+1}\cot^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}}{1+a^2x^2}\,dx$

459. $\int \frac{\sin^{-1}(ax)}{x^2}\,dx = a\log\left(\frac{1-\sqrt{1-a^2x^2}}{x}\right) - \frac{\sin^{-1}(ax)}{x}$

460. $\int \frac{\cos^{-1}(ax)\,dx}{x^2} = -\frac{1}{x}\cos^{-1}(ax) + a\log\frac{1+\sqrt{1-a^2x^2}}{x}$

461. $\int \frac{\tan^{-1}(ax)\,dx}{x^2} = -\frac{1}{x}\tan^{-1}(ax) - \frac{a}{2}\log\frac{1+a^2x^2}{x^2}$

462. $\int \frac{\cot^{-1} ax}{x^2}\,dx = -\frac{1}{x}\cot^{-1} ax - \frac{a}{2}\log\frac{x^2}{a^2x^2+1}$

463. $\int (\sin^{-1} ax)^2\,dx = x(\sin^{-1} ax)^2 - 2x + \frac{2\sqrt{1-a^2x^2}}{a}\sin^{-1} ax$

464. $\int (\cos^{-1} ax)^2\,dx = x(\cos^{-1} ax)^2 - 2x - \frac{2\sqrt{1-a^2x^2}}{a}\cos^{-1} ax$

465. $$\int (\sin^{-1} ax)^n\,dx = \begin{cases} x(\sin^{-1} ax)^n + \frac{n\sqrt{1-a^2x^2}}{a}(\sin^{-1} ax)^{n-1} \\ \qquad -n(n-1)\int (\sin^{-1} ax)^{n-2}\,dx \\ \text{or} \\ \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{n!}{(n-2r)!} x(\sin^{-1} ax)^{n-2r} \\ \qquad + \sum_{r=0}^{[\frac{n-1}{2}]} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\sin^{-1} ax)^{n-2r-1} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$. Thus $[3.5]$ means 3; $[5] = 5$, $[\frac{1}{2}] = 0$.

INTEGRALS (Continued)

466. $$\int (\cos^{-1} ax)^n\,dx = \begin{cases} x(\cos^{-1} ax)^n - \dfrac{n\sqrt{1-a^2x^2}}{a}(\cos^{-1} ax)^{n-1} \\ \qquad - n(n-1)\displaystyle\int (\cos^{-1} ax)^{n-2}\,dx \\ \text{or} \\ \displaystyle\sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{n!}{(n-2r)!} x(\cos^{-1} ax)^{n-2r} \\ \qquad - \displaystyle\sum_{r=0}^{\left[\frac{n-1}{2}\right]} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!\,a}(\cos^{-1} ax)^{n-2r-1} \end{cases}$$

467. $$\int \frac{1}{\sqrt{1-a^2x^2}}(\sin^{-1} ax)\,dx = \frac{1}{2a}(\sin^{-1} ax)^2$$

468. $$\int \frac{x^n}{\sqrt{1-a^2x^2}}(\sin^{-1} ax)\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\sin^{-1} ax + \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}}{\sqrt{1-a^2x^2}}\sin^{-1} ax\,dx$$

469. $$\int \frac{1}{\sqrt{1-a^2x^2}}(\cos^{-1} ax)\,dx = -\frac{1}{2a}(\cos^{-1} ax)^2$$

470. $$\int \frac{x^n}{\sqrt{1-a^2x^2}}(\cos^{-1} ax)\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\cos^{-1} ax - \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}}{\sqrt{1-a^2x^2}}\cos^{-1} ax\,dx$$

471. $$\int \frac{\tan^{-1} ax}{a^2x^2+1}\,dx = \frac{1}{2a}(\tan^{-1} ax)^2$$

472. $$\int \frac{\cot^{-1} ax}{a^2x^2+1}\,dx = -\frac{1}{2a}(\cot^{-1} ax)^2$$

473. $$\int x\sec^{-1} ax\,dx = \frac{x^2}{2}\sec^{-1} ax - \frac{1}{2a^2}\sqrt{a^2x^2-1}$$

474. $$\int x^n\sec^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sec^{-1} ax - \frac{1}{n+1}\int \frac{x^n\,dx}{\sqrt{a^2x^2-1}}$$

475. $$\int \frac{\sec^{-1} ax}{x^2}\,dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}$$

476. $$\int x\csc^{-1} ax\,dx = \frac{x^2}{2}\csc^{-1} ax + \frac{1}{2a^2}\sqrt{a^2x^2-1}$$

477. $$\int x^n\csc^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\csc^{-1} ax + \frac{1}{n+1}\int \frac{x^n\,dx}{\sqrt{a^2x^2-1}}$$

INTEGRALS (Continued)

78. $$\int \frac{\csc^{-1} ax}{x^2}\,dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2x^2 - 1}}{x}$$

FORMS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

79. $$\int f(\sin x)\,dx = 2\int f\left(\frac{2z}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(z = \tan\frac{x}{2}\right)$$

80. $$\int f(\cos x)\,dx = 2\int f\left(\frac{1-z^2}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(z = \tan\frac{x}{2}\right)$$

81. $$\int f(\sin x)\,dx = \int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad (u = \sin x)$$

82. $$\int f(\cos x)\,dx = -\int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad (u = \cos x)$$

83. $$\int f(\sin x, \cos x)\,dx = \int f(u, \sqrt{1-u^2})\frac{du}{\sqrt{1-u^2}}, \qquad (u = \sin x)$$

84. $$\int f(\sin x, \cos x)\,dx = 2\int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(z = \tan\frac{x}{2}\right)$$

LOGARITHMIC FORMS

85. $$\int (\log x)\,dx = x\log x - x$$

86. $$\int x(\log x)\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$$

87. $$\int x^2(\log x)\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9}$$

88. $$\int x^n(\log ax)\,dx = \frac{x^{n+1}}{n+1}\log ax - \frac{x^{n+1}}{(n+1)^2}$$

89. $$\int (\log x)^2\,dx = x(\log x)^2 - 2x\log x + 2x$$

90. $$\int (\log x)^n\,dx = \begin{cases} x(\log x)^n - n\int (\log x)^{n-1}\,dx, & (n \neq -1) \\ \text{or} & \\ (-1)^n n!\,x \sum_{r=0}^{n} \frac{(-\log x)^r}{r!} & \end{cases}$$

The square roots appearing in these formulas may be plus or minus, depending on the quadrant of x. Care must be used to give them the proper sign.

INTEGRALS (Continued)

491. $\int \frac{(\log x)^n}{x}\,dx = \frac{1}{n+1}(\log x)^{n+1}$

492. $\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2\cdot 2!} + \frac{(\log x)^3}{3\cdot 3!} + \cdots$

493. $\int \frac{dx}{x\log x} = \log(\log x)$

494. $\int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}$

495. $\int \frac{x^m\,dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\log x)^{n-1}}$

496. $\int x^m(\log x)^n\,dx = \begin{cases} \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m(\log x)^{n-1}\,dx \\ \text{or} \\ (-1)^n\frac{n!}{m+1}x^{m+1}\sum_{r=0}^{n}\frac{(-\log x)^r}{r!(m+1)^{n-r}} \end{cases}$

497. $\int x^p\cos(b\ln x)\,dx = \frac{x^{p+1}}{(p+1)^2+b^2}\cdot[b\sin(b\ln x) + (p+1)\cos(b\ln x)] + c$

498. $\int x^p\sin(b\ln x)\,dx = \frac{x^{p+1}}{(p+1)^2+b^2}\cdot[(p+1)\sin(b\ln x) - b\cos(b\ln x)] + c$

499. $\int [\log(ax+b)]\,dx = \frac{ax+b}{a}\log(ax+b) - x$

500. $\int \frac{\log(ax+b)}{x^2}\,dx = \frac{a}{b}\log x - \frac{ax+b}{bx}\log(ax+b)$

501. $\int x^m[\log(ax+b)]\,dx = \frac{1}{m+1}\left[x^{m+1} - \left(-\frac{b}{a}\right)^{m+1}\right]\log(ax+b) - \frac{1}{m+1}\left(-\frac{b}{a}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(-\frac{ax}{b}\right)^r$

502. $\int \frac{\log(ax+b)}{x^m}\,dx = -\frac{1}{m-1}\frac{\log(ax+b)}{x^{m-1}} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\log\frac{ax+b}{x} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(-\frac{b}{ax}\right)^r, (m > 2)$

503. $\int \left[\log\frac{x+a}{x-a}\right]dx = (x+a)\log(x+a) - (x-a)\log(x-a)$

504. $\int x^m\left[\log\frac{x+a}{x-a}\right]dx = \frac{x^{m+1}-(-a)^{m+1}}{m+1}\log(x+a) - \frac{x^{m+1}-a^{m+1}}{m+1}\log(x-a) + \frac{2a^{m+1}}{m+1}\sum_{r=1}^{\left[\frac{m+1}{2}\right]}\frac{1}{m-2r+2}\left(\frac{x}{a}\right)^{m-2r+2}$

See note integral 392.

INTEGRALS (Continued)

5. $\int \frac{1}{x^2}\left[\log \frac{x+a}{x-a}\right] dx = \frac{1}{x}\log\frac{x-a}{x+a} - \frac{1}{a}\log\frac{x^2-a^2}{x^2}$

6. $$\int (\log X)\, dx = \begin{cases} \left(x + \frac{b}{2c}\right)\log X - 2x + \frac{\sqrt{4ac-b^2}}{c}\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}}, & (b^2 - 4ac < 0) \\ \text{or} & \\ \left(x + \frac{b}{2c}\right)\log X - 2x + \frac{\sqrt{b^2-4ac}}{c}\tanh^{-1}\frac{2cx+b}{\sqrt{b^2-4ac}}, & (b^2 - 4ac > 0) \\ \text{where} & \\ X = a + bx + cx^2 & \end{cases}$$

7. $\int x^n(\log X)\, dx = \frac{x^{n+1}}{n+1}\log X - \frac{2c}{n+1}\int \frac{x^{n+2}}{X}\, dx - \frac{b}{n+1}\int \frac{x^{n+1}}{X}\, dx$

where $X = a + bx + cx^2$

8. $\int [\log(x^2 + a^2)]\, dx = x\log(x^2 + a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$

9. $\int [\log(x^2 - a^2)]\, dx = x\log(x^2 - a^2) - 2x + a\log\frac{x+a}{x-a}$

0. $\int x[\log(x^2 \pm a^2)]\, dx = \frac{1}{2}(x^2 \pm a^2)\log(x^2 \pm a^2) - \frac{1}{2}x^2$

1. $\int [\log(x + \sqrt{x^2 \pm a^2})]\, dx = x\log(x + \sqrt{x^2 \pm a^2}) - \sqrt{x^2 \pm a^2}$

2. $\int x[\log(x + \sqrt{x^2 \pm a^2})]\, dx = \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right)\log(x + \sqrt{x^2 \pm a^2}) - \frac{x\sqrt{x^2 \pm a^2}}{4}$

3. $$\int x^m[\log(x + \sqrt{x^2 \pm a^2})]\, dx = \frac{x^{m+1}}{m+1}\log(x + \sqrt{x^2 \pm a^2}) - \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2 \pm a^2}}\, dx$$

4. $\int \frac{\log(x + \sqrt{x^2 + a^2})}{x^2}\, dx = -\frac{\log(x + \sqrt{x^2 + a^2})}{x} - \frac{1}{a}\log\frac{a + \sqrt{x^2 + a^2}}{x}$

5. $\int \frac{\log(x + \sqrt{x^2 - a^2})}{x^2}\, dx = -\frac{\log(x + \sqrt{x^2 - a^2})}{x} + \frac{1}{|a|}\sec^{-1}\frac{x}{a}$

INTEGRALS (Continued)

516. $\int x^n \log(x^2 - a^2)\,dx = \frac{1}{n+1}\left[x^{n+1}\log(x^2 - a^2) - a^{n+1}\log(x-a) - (-a)^{n+1}\log(x+a) - 2\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{a^{2r}x^{n-2r+1}}{n-2r+1}\right]$

See note integral 392.

EXPONENTIAL FORMS

517. $\int e^x\,dx = e^x$

518. $\int e^{-x}\,dx = -e^{-x}$

519. $\int e^{ax}\,dx = \frac{e^{ax}}{a}$

520. $\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$

521. $\int x^m e^{ax}\,dx = \begin{cases} \frac{x^m e^{ax}}{a} - \frac{m}{a}\int x^{m-1} e^{ax}\,dx \\ \text{or} \\ e^{ax}\sum_{r=0}^{m}(-1)^r \frac{m!x^{m-r}}{(m-r)!a^{r+1}} \end{cases}$

522. $\int \frac{e^{ax}\,dx}{x} = \log x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots$

523. $\int \frac{e^{ax}}{x^m}\,dx = -\frac{1}{m-1}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{e^{ax}}{x^{m-1}}\,dx$

524. $\int e^{ax}\log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$

525. $\int \frac{dx}{1+e^x} = x - \log(1+e^x) = \log\frac{e^x}{1+e^x}$

526. $\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\log(a+be^{px})$

527. $\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right), \qquad (a > 0, b > 0)$

528. $\int \frac{dx}{ae^{mx}-be^{-mx}} = \begin{cases} \frac{1}{2m\sqrt{ab}}\log\frac{\sqrt{a}\,e^{mx}-\sqrt{b}}{\sqrt{a}\,e^{mx}+\sqrt{b}} \\ \text{or} \\ \frac{-1}{m\sqrt{ab}}\tanh^{-1}\left(\sqrt{\frac{a}{b}}\,e^{mx}\right), \qquad (a > 0, b > 0) \end{cases}$

INTEGRALS (Continued)

529. $\displaystyle\int (a^x - a^{-x})\,dx = \frac{a^x + a^{-x}}{\log a}$

530. $\displaystyle\int \frac{e^{ax}}{b + ce^{ax}}\,dx = \frac{1}{ac}\log(b + ce^{ax})$

531. $\displaystyle\int \frac{x\,e^{ax}}{(1 + ax)^2}\,dx = \frac{e^{ax}}{a^2(1 + ax)}$

532. $\displaystyle\int x\,e^{-x^2}\,dx = -\tfrac{1}{2}\,e^{-x^2}$

533. $\displaystyle\int e^{ax}[\sin(bx)]\,dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2 + b^2}$

534. $\displaystyle\int e^{ax}[\sin(bx)][\sin(cx)]\,dx = \frac{e^{ax}[(b - c)\sin(b - c)x + a\cos(b - c)x]}{2[a^2 + (b - c)^2]} - \frac{e^{ax}[(b + c)\sin(b + c)x + a\cos(b + c)x]}{2[a^2 + (b + c)^2]}$

535. $$\int e^{ax}[\sin(bx)][\cos(cx)]\,dx = \begin{cases} \dfrac{e^{ax}[a\sin(b - c)x - (b - c)\cos(b - c)x]}{2[a^2 + (b - c)^2]} + \dfrac{e^{ax}[a\sin(b + c)x - (b + c)\cos(b + c)x]}{2[a^2 + (b + c)^2]} \\ \text{or} \\ \dfrac{e^{ax}}{\rho}[(a\sin bx - b\cos bx)[\cos(cx - \alpha)] - c(\sin bx)\sin(cx - \alpha)] \\ \text{where} \\ \rho = \sqrt{(a^2 + b^2 - c^2)^2 + 4a^2c^2}, \\ \rho\cos\alpha = a^2 + b^2 - c^2, \qquad \rho\sin\alpha = 2ac \end{cases}$$

536. $$\int e^{ax}[\sin(bx)][\sin(bx + c)]\,dx = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx + c) + 2b\sin(2bx + c)]}{2(a^2 + 4b^2)}$$

537. $$\int e^{ax}[\sin(bx)][\cos(bx + c)]\,dx = \frac{-e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx + c) - 2b\cos(2bx + c)]}{2(a^2 + 4b^2)}$$

538. $\displaystyle\int e^{ax}[\cos(bx)]\,dx = \frac{e^{ax}}{a^2 + b^2}[a\cos(bx) + b\sin(bx)]$

INTEGRALS (Continued)

539. $$\int e^{ax}[\cos(bx)][\cos(cx)]\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$$

540. $$\int e^{ax}[\cos(bx)][\cos(bx+c)]\,dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$$

541. $$\int e^{ax}[\cos(bx)][\sin(bx+c)]\,dx = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$$

542. $$\int e^{ax}[\sin^n bx]\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)\,e^{ax}\sin^{n-1}bx + n(n-1)b^2\int e^{ax}[\sin^{n-2}bx]\,dx\right]$$

543. $$\int e^{ax}[\cos^n bx]\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1}bx + n(n-1)b^2\int e^{ax}[\cos^{n-2}bx]\,dx\right]$$

544. $$\int x^m e^x \sin x\,dx = \frac{1}{2}x^m e^x(\sin x - \cos x) - \frac{m}{2}\int x^{m-1}e^x\sin x\,dx + \frac{m}{2}\int x^{m-1}e^x\cos x\,dx$$

545. $$\int x^m e^{ax}[\sin bx]\,dx = \begin{cases} x^m e^{ax}\dfrac{a\sin bx - b\cos bx}{a^2+b^2} - \dfrac{m}{a^2+b^2}\displaystyle\int x^{m-1}e^{ax}(a\sin bx - b\cos bx)\,dx \\ \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}\frac{(-1)^r m!\,x^{m-r}}{\rho^{r+1}(m-r)!}\sin[bx-(r+1)\alpha] \\ \text{where} \\ \rho = \sqrt{a^2+b^2}, \quad \rho\cos\alpha = a, \quad \rho\sin\alpha = b \end{cases}$$

546. $$\int x^m e^x\cos x\,dx = \frac{1}{2}x^m e^x(\sin x + \cos x) - \frac{m}{2}\int x^{m-1}e^x\sin x\,dx - \frac{m}{2}\int x^{m-1}e^x\cos x\,dx$$

INTEGRALS (Continued)

7. $$\int x^m e^{ax}\cos bx\,dx = \begin{cases} x^m e^{ax}\dfrac{a\cos bx + b\sin bx}{a^2+b^2} \\ \qquad -\dfrac{m}{a^2+b^2}\displaystyle\int x^{m-1}e^{ax}(a\cos bx + b\sin bx)\,dx \\ \qquad \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}\dfrac{(-1)^r m!\,x^{m-r}}{\rho^{r+1}(m-r)!}\cos[bx-(r+1)\alpha] \\ \qquad \text{where} \\ \rho = \sqrt{a^2+b^2}, \qquad \rho\cos\alpha = a, \qquad \rho\sin\alpha = b \end{cases}$$

8. $$\int e^{ax}(\cos^m x)(\sin^n x)\,dx = \begin{cases} \dfrac{e^{ax}\cos^{m-1}x\sin^n x[a\cos x + (m+n)\sin x]}{(m+n)^2+a^2} \\ \qquad -\dfrac{na}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)\,dx \\ \qquad +\dfrac{(m-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^n x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}\cos^m x\sin^{n-1}x[a\sin x - (m+n)\cos x]}{(m+n)^2+a^2} \\ \qquad +\dfrac{ma}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)\,dx \\ \qquad +\dfrac{(n-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)(a\sin x\cos x + m\sin^2 x - n\cos^2 x)}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^n x)\,dx \\ \qquad +\dfrac{n(n-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)(a\cos x\sin x + m\sin^2 x - n\cos^2 x)}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^{n-2}x)\,dx \\ \qquad +\dfrac{(n-m)(n+m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \end{cases}$$

INTEGRALS (Continued)

549. $$\int x\, e^{ax}(\sin bx)\, dx = \frac{x\, e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2)\sin bx - 2ab \cos bx]$$

550. $$\int x\, e^{ax}(\cos bx)\, dx = \frac{x\, e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2)\cos bx + 2ab \sin bx]$$

551. $$\int \frac{e^{ax}}{\sin^n x}\, dx = -\frac{e^{ax}[a \sin x + (n-2)\cos x]}{(n-1)(n-2)\sin^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\sin^{n-2} x}\, dx$$

552. $$\int \frac{e^{ax}}{\cos^n x}\, dx = -\frac{e^{ax}[a \cos x - (n-2)\sin x]}{(n-1)(n-2)\cos^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\cos^{n-2} x}\, dx$$

553. $$\int e^{ax} \tan^n x\, dx = e^{ax}\frac{\tan^{n-1} x}{n-1} - \frac{a}{n-1}\int e^{ax} \tan^{n-1} x\, dx - \int e^{ax} \tan^{n-2} x\, dx$$

HYPERBOLIC FORMS

554. $$\int (\sinh x)\, dx = \cosh x$$

555. $$\int (\cosh x)\, dx = \sinh x$$

556. $$\int (\tanh x)\, dx = \log \cosh x$$

557. $$\int (\coth x)\, dx = \log \sinh x$$

558. $$\int (\text{sech}\, x)\, dx = \tan^{-1}(\sinh x)$$

559. $$\int \text{csch}\, x\, dx = \log \tanh\left(\frac{x}{2}\right)$$

560. $$\int x(\sinh x)\, dx = x \cosh x - \sinh x$$

561. $$\int x^n(\sinh x)\, dx = x^n \cosh x - n \int x^{n-1}(\cosh x)\, dx$$

562. $$\int x(\cosh x)\, dx = x \sinh x - \cosh x$$

563. $$\int x^n(\cosh x)\, dx = x^n \sinh x - n \int x^{n-1}(\sinh x)\, dx$$

INTEGRALS (Continued)

564. $\int (\operatorname{sech} x)(\tanh x)\,dx = -\operatorname{sech} x$

565. $\int (\operatorname{csch} x)(\coth x)\,dx = -\operatorname{csch} x$

566. $\int (\sinh^2 x)\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

567. $$\int (\sinh^m x)(\cosh^n x)\,dx = \begin{cases} \frac{1}{m+n}(\sinh^{m+1} x)(\cosh^{n-1} x) \\ \qquad + \frac{n-1}{m+n}\int (\sinh^m x)(\cosh^{n-2} x)\,dx \\ \text{or} \\ \frac{1}{m+n}\sinh^{m-1} x \cosh^{n+1} x \\ - \frac{m-1}{m+n}\int (\sinh^{m-2} x)(\cosh^n x)\,dx, \quad (m+n \neq 0) \end{cases}$$

568. $$\int \frac{dx}{(\sinh^m x)(\cosh^n x)} = \begin{cases} -\frac{1}{(m-1)(\sinh^{m-1} x)(\cosh^{n-1} x)} \\ \qquad - \frac{m+n-2}{m-1}\int \frac{dx}{(\sinh^{m-2} x)(\cosh^n x)}, \quad (m \neq 1) \\ \text{or} \\ \frac{1}{(n-1)\sinh^{m-1} x \cosh^{n-1} x} \\ \qquad + \frac{m+n-2}{n-1}\int \frac{dx}{(\sinh^m x)(\cosh^{n-2} x)}, \quad (n \neq 1) \end{cases}$$

569. $\int (\tanh^2 x)\,dx = x - \tanh x$

570. $\int (\tanh^n x)\,dx = -\frac{\tanh^{n-1} x}{n-1} + \int (\tanh^{n-2} x)\,dx, \quad (n \neq 1)$

571. $\int (\operatorname{sech}^2 x)\,dx = \tanh x$

572. $\int (\cosh^2 x)\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

573. $\int (\coth^2 x)\,dx = x - \coth x$

574. $\int (\coth^n x)\,dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x\,dx, \quad (n \neq 1)$

INTEGRALS (Continued)

575. $\int (\operatorname{csch}^2 x)\, dx = -\operatorname{ctnh} x$

576. $\int (\sinh mx)(\sinh nx)\, dx = \frac{\sinh (m+n)x}{2(m+n)} - \frac{\sinh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

577. $\int (\cosh mx)(\cosh nx)\, dx = \frac{\sinh (m+n)x}{2(m+n)} + \frac{\sinh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

578. $\int (\sinh mx)(\cosh nx)\, dx = \frac{\cosh (m+n)x}{2(m+n)} + \frac{\cosh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

579. $\int \left(\sinh^{-1} \frac{x}{a}\right) dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2 + a^2}, \qquad (a > 0)$

580. $\int x\left(\sinh^{-1} \frac{x}{a}\right) dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right) \sinh^{-1} \frac{x}{a} - \frac{x}{4}\sqrt{x^2 + a^2}, \qquad (a > 0)$

581. $\int x^n(\sinh^{-1} x)\, dx = \frac{x^{n+1}}{n+1} \sinh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(1+x^2)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

582. $\int \left(\cosh^{-1} \frac{x}{a}\right) dx = \begin{cases} x \cosh^{-1} \frac{x}{a} - \sqrt{x^2 - a^2}, & \left(\cosh^{-1} \frac{x}{a} > 0\right) \\ \text{or} & \\ x \cosh^{-1} \frac{x}{a} + \sqrt{x^2 - a^2}, & \left(\cosh^{-1} \frac{x}{a} < 0\right), \end{cases} \qquad (a > 0)$

583. $\int x\left(\cosh^{-1} \frac{x}{a}\right) dx = \frac{2x^2 - a^2}{4} \cosh^{-1} \frac{x}{a} - \frac{x}{4}(x^2 - a^2)^{\frac{1}{2}}$

584. $\int x^n(\cosh^{-1} x)\, dx = \frac{x^{n+1}}{n+1} \cosh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{(x^2-1)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

585. $\int \left(\tanh^{-1} \frac{x}{a}\right) dx = x \tanh^{-1} \frac{x}{a} + \frac{a}{2} \log (a^2 - x^2), \qquad \left(\left|\frac{x}{a}\right| < 1\right)$

586. $\int \left(\coth^{-1} \frac{x}{a}\right) dx = x \coth^{-1} \frac{x}{a} + \frac{a}{2} \log (x^2 - a^2), \qquad \left(\left|\frac{x}{a}\right| > 1\right)$

587. $\int x\left(\tanh^{-1} \frac{x}{a}\right) dx = \frac{x^2 - a^2}{2} \tanh^{-1} \frac{x}{a} + \frac{ax}{2}, \qquad \left(\left|\frac{x}{a}\right| < 1\right)$

588. $\int x^n\left(\tanh^{-1} x\right) dx = \frac{x^{n+1}}{n+1} \tanh^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1 - x^2}\, dx, \qquad (n \neq -1)$

589. $\int x\left(\coth^{-1} \frac{x}{a}\right) dx = \frac{x^2 - a^2}{2} \coth^{-1} \frac{x}{a} + \frac{ax}{2}, \qquad \left(\left|\frac{x}{a}\right| > 1\right)$

590. $\int x^n(\coth^{-1} x)\, dx = \frac{x^{n+1}}{n+1} \coth^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{x^2 - 1}\, dx, \qquad (n \neq -1)$

INTEGRALS (Continued)

1. $\int (\operatorname{sech}^{-1} x)\, dx = x \operatorname{sech}^{-1} x + \sin^{-1} x$

2. $\int x \operatorname{sech}^{-1} x\, dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1 - x^2}$

3. $\int x^n \operatorname{sech}^{-1} x\, dx = \frac{x^{n+1}}{n+1} \operatorname{sech}^{-1} x + \frac{1}{n+1} \int \frac{x^n}{(1 - x^2)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

4. $\int \operatorname{csch}^{-1} x\, dx = x \operatorname{csch}^{-1} x + \frac{x}{|x|} \sinh^{-1} x$

5. $\int x \operatorname{csch}^{-1} x\, dx = \frac{x^2}{2} \operatorname{csch}^{-1} x + \frac{1}{2} \frac{x}{|x|}\sqrt{1 + x^2}$

6. $\int x^n \operatorname{csch}^{-1} x\, dx = \frac{x^{n+1}}{n+1} \operatorname{csch}^{-1} x + \frac{1}{n+1} \frac{x}{|x|} \int \frac{x^n}{(x^2 + 1)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

DEFINITE INTEGRALS

7. $\int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \frac{1}{n} \prod_{m=1}^{\infty} \frac{\left(1 + \frac{1}{m}\right)^n}{1 + \frac{n}{m}}$

$= \Gamma(n), n \neq 0, -1, -2, -3, \ldots \qquad$ (Gamma Function)

98. $\int_0^\infty t^n p^{-t}\, dt = \frac{n!}{(\log p)^{n+1}}, \qquad (n = 0, 1, 2, 3, \ldots \text{ and } p > 0)$

99. $\int_0^\infty t^{n-1} e^{-(a+1)t}\, dt = \frac{\Gamma(n)}{(a+1)^n}, \qquad (n > 0, a > -1)$

00. $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \qquad (m > -1, n > -1)$

01. $\Gamma(n)$ is finite if $n > 0$, $\Gamma(n+1) = n\Gamma(n)$

02. $\Gamma(n) \cdot \Gamma(1 - n) = \frac{\pi}{\sin n\pi}$

03. $\Gamma(n) = (n-1)!$ if $n =$ integer > 0

04. $\Gamma(\frac{1}{2}) = 2 \int_0^\infty e^{-t^2}\, dt = \sqrt{\pi} = 1.7724538509 \cdots = (-\frac{1}{2})!$

05. $\Gamma(n + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n} \sqrt{\pi} \qquad n = 1, 2, 3, \ldots$

06. $\Gamma(-n + \frac{1}{2}) = \frac{(-1)^n 2^n \sqrt{\pi}}{1 \cdot 3 \cdot 5 \ldots (2n-1)} \qquad n = 1, 2, 3, \ldots$

DEFINITE INTEGRALS (Continued)

607. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$

(Beta functio

608. $B(m,n) = B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where m and n are any positive real numbers.

609. $\int_a^b (x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}\frac{\Gamma(m+1)\cdot\Gamma(n+1)}{\Gamma(m+n+2)}$,

$(m > -1, n > -1, b >$

610. $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \qquad [m > 1]$

611. $\int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \qquad [p < 1]$

612. $\int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \qquad [p < 1]$

613. $\int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi}$

$= B(p, 1-p) = \Gamma(p)\Gamma(1-p), \qquad [0 < p < 1]$

614. $\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n} = \frac{\pi}{n\sin\frac{m\pi}{n}}, \qquad [0 < m < n]$

615. $\int_0^\infty \frac{x^a\,dx}{(m+x^b)^c} = \frac{m^{\frac{a+1-bc}{b}}}{b}\left[\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c-\frac{a+1}{b}\right)}{\Gamma(c)}\right]$

$\left(a > -1, b > 0, m > 0, c >\right.$

616. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$

617. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a > 0$; 0, if $a = 0$; $-\frac{\pi}{2}$, if $a < 0$

618. $\int_0^a (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{1}{2}\int_{-a}^a (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{1\cdot3\cdot5\ldots n}{2\cdot4\cdot6\ldots(n+1)}\cdot\frac{\pi}{2}\cdot a^{n+1}$ (n odd)

619. $\int_0^a x^m(a^2-x^2)^{\frac{n}{2}}\,dx = \begin{cases} \frac{1}{2}a^{m+n+1}B\left(\frac{m+1}{2}, \frac{n+2}{2}\right) \\ \text{or} \\ \frac{1}{2}a^{m+n+1}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)} \end{cases}$

DEFINITE INTEGRALS (Continued)

620. $$\int_0^{\pi/2} (\sin^n x)\,dx = \begin{cases} \displaystyle\int_0^{\pi/2} (\cos^n x)\,dx \\ \text{or} \\ \dfrac{1\cdot 3\cdot 5\cdot 7\ldots(n-1)}{2\cdot 4\cdot 6\cdot 8\ldots(n)}\dfrac{\pi}{2}, & (n \text{ an even integer}, n \neq 0) \\ \text{or} \\ \dfrac{2\cdot 4\cdot 6\cdot 8\ldots(n-1)}{1\cdot 3\cdot 5\cdot 7\ldots(n)}, & (n \text{ an odd integer}, n \neq 1) \\ \text{or} \\ \dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, & (n > -1) \end{cases}$$

621. $$\int_0^{\infty} \frac{\sin mx\,dx}{x} = \frac{\pi}{2}, \text{ if } m > 0;\ 0, \text{ if } m = 0;\ -\frac{\pi}{2}, \text{ if } m < 0$$

622. $$\int_0^{\infty} \frac{\cos x\,dx}{x} = \infty$$

623. $$\int_0^{\infty} \frac{\tan x\,dx}{x} = \frac{\pi}{2}$$

624. $$\int_0^{\pi} \sin ax \cdot \sin bx\,dx = \int_0^{\pi} \cos ax \cdot \cos bx\,dx = 0, \qquad (a \neq b;\ a, b \text{ integers})$$

625. $$\int_0^{\pi/a} [\sin(ax)][\cos(ax)]\,dx = \int_0^{\pi} [\sin(ax)][\cos(ax)]\,dx = 0$$

626. $$\int_0^{\pi} [\sin(ax)][\cos(bx)]\,dx = \frac{2a}{a^2 - b^2}, \text{ if } a - b \text{ is odd, or } 0 \text{ if } a - b \text{ is even}$$

627. $$\int_0^{\infty} \frac{\sin x \cos mx\,dx}{x}$$

$$= 0, \text{ if } m < -1 \text{ or } m > 1;\ \frac{\pi}{4}, \text{ if } m = \pm 1;\ \frac{\pi}{2}, \text{ if } m^2 < 1$$

628. $$\int_0^{\infty} \frac{\sin ax \sin bx}{x^2}\,dx = \frac{\pi a}{2}, \qquad (a \leq b)$$

629. $$\int_0^{\pi} \sin^2 mx\,dx = \int_0^{\pi} \cos^2 mx\,dx = \frac{\pi}{2}$$

630. $$\int_0^{\infty} \frac{\sin^2(px)}{x^2}\,dx = \frac{\pi p}{2}$$

DEFINITE INTEGRALS (Continued)

631. $\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0 < p < 1$

632. $\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0 < p < 1$

633. $\int_0^\infty \frac{1 - \cos px}{x^2}\,dx = \frac{\pi p}{2}$

634. $\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \left\{0, \quad q > p > 0; \quad \frac{\pi}{2}, \quad p > q > 0; \quad \frac{\pi}{4}, \quad p = q > 0\right\}$

635. $\int_0^\infty \frac{\cos(mx)}{x^2 + a^2}\,dx = \frac{\pi}{2|a|} e^{-|ma|}$

636. $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

637. $\int_0^\infty \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \qquad n > 1$

638. $\int_0^\infty \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \qquad n > 1$

639. $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$

640. (a) $\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$ (b) $\int_0^\infty \frac{\sin^3 x}{x^2}\,dx \frac{3}{4}\log 3$

641. $\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

642. $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

643. $\int_0^{\pi/2} \frac{dx}{1 + a\cos x} = \frac{\cos^{-1} a}{\sqrt{1 - a^2}}, \qquad (a < 1)$

644. $\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}, \qquad (a > b \geq 0)$

645. $\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1 - a^2}}, \qquad (a^2 < 1)$

646. $\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}$

647. $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}$

DEFINITE INTEGRALS (Continued)

648. $$\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3b^3}, \qquad (a, b > 0)$$

649. $$\int_0^{\pi/2} \sin^{n-1} x \cos^{m-1} x \, dx = \frac{1}{2} \mathrm{B}\left(\frac{n}{2}, \frac{m}{2}\right), \qquad m \text{ and } n \text{ positive integers}$$

650. $$\int_0^{\pi/2} (\sin^{2n+1} \theta) \, d\theta = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{1 \cdot 3 \cdot 5 \ldots (2n+1)}, \qquad (n = 1, 2, 3 \ldots)$$

651. $$\int_0^{\pi/2} (\sin^{2n} \theta) \, d\theta = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \ldots (2n)} \left(\frac{\pi}{2}\right), \qquad (n = 1, 2, 3 \ldots)$$

652. $$\int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\}$$

653. $$\int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$$

654. $$\int_0^{\pi/2} \sqrt{\cos \theta} \, d\theta = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2}$$

655. $$\int_0^{\pi/2} (\tan^h \theta) \, d\theta = \frac{\pi}{2 \cos\left(\frac{h\pi}{2}\right)}, \qquad (0 < h < 1)$$

656. $$\int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} \, dx = \frac{\pi}{2} \log \frac{a}{b}, \qquad (a, b > 0)$$

657. The area enclosed by a curve defined through the equation $x^{\frac{b}{c}} + y^{\frac{b}{c}} = a^{\frac{b}{c}}$ where $a > 0$, c a positive odd integer and b a positive even integer is given by

$$\frac{\left[\Gamma\left(\frac{c}{b}\right)\right]^2}{\Gamma\left(\frac{2c}{b}\right)} \left(\frac{2ca^2}{b}\right)$$

658. $I = \iiint\limits_R x^{h-1} y^{m-1} z^{n-1} \, dv$, where R denotes the region of space bounded by the co-ordinate planes and that portion of the surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^k = 1$, which lies in the first octant and where $h, m, n, p, q, k, a, b, c$, denote positive real numbers is given by

$$\int_0^a x^{h-1} \, dx \int_0^{b\left[1-\left(\frac{x}{a}\right)^p\right]^{\frac{1}{q}}} y^m \, dy \int_0^{c\left[1-\left(\frac{x}{a}\right)^p-\left(\frac{y}{b}\right)^q\right]^{\frac{1}{k}}} z^{n-1} \, dz$$

$$= \frac{a^h b^m c^n}{pqk} \frac{\Gamma\left(\frac{h}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{k}\right)}{\Gamma\left(\frac{h}{p} + \frac{m}{q} + \frac{n}{k} + 1\right)}$$

DEFINITE INTEGRALS (Continued)

659. $$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \qquad (a > 0)$$

660. $$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\frac{b}{a}, \qquad (a, b > 0)$$

661. $$\int_0^\infty x^n e^{-ax}\,dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ \text{or} & \\ \dfrac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$$

662. $$\int_0^\infty x^n \exp(-ax^p)\,dx = \frac{\Gamma(k)}{pa^k}, \qquad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p}\right)$$

663. $$\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right), \qquad (a > 0)$$

664. $$\int_0^\infty x\,e^{-x^2}\,dx = \tfrac{1}{2}$$

665. $$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$$

666. $$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

667. $$\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}, \qquad (a > 0)$$

668. $$\int_0^1 x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}\left[1 - e^{-a}\sum_{r=0}^{m}\frac{a^r}{r!}\right]$$

669. $$\int_0^\infty e^{\left(-x^2 - \frac{a^2}{x^2}\right)}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}, \qquad (a \geq 0)$$

670. $$\int_0^\infty e^{-nx}\sqrt{x}\,dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$$

671. $$\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}$$

672. $$\int_0^\infty e^{-ax}(\cos mx)\,dx = \frac{a}{a^2 + m^2}, \qquad (a > 0)$$

673. $$\int_0^\infty e^{-ax}(\sin mx)\,dx = \frac{m}{a^2 + m^2}, \qquad (a > 0)$$

DEFINITE INTEGRALS (Continued)

74. $\int_0^\infty x\, e^{-ax}[\sin(bx)]\, dx = \frac{2ab}{(a^2+b^2)^2}, \qquad (a > 0)$

75. $\int_0^\infty x\, e^{-ax}[\cos(bx)]\, dx = \frac{a^2-b^2}{(a^2+b^2)^2}, \qquad (a > 0)$

76. $\int_0^\infty x^n e^{-ax}[\sin(bx)]\, dx = \frac{n![(a+ib)^{n+1}-(a-ib)^{n+1}]}{2i(a^2+b^2)^{n+1}}, \qquad (i^2 = -1, a > 0)$

77. $\int_0^\infty x^n e^{-ax}[\cos(bx)]\, dx = \frac{n![(a-ib)^{n+1}+(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}, \qquad (i^2 = -1, a > 0)$

78. $\int_0^\infty \frac{e^{-ax}\sin x}{x}\, dx = \cot^{-1} a, \qquad (a > 0)$

79. $\int_0^\infty e^{-a^2x^2}\cos bx\, dx = \frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right), \qquad (ab \neq 0)$

80. $\int_0^\infty e^{-t\cos\phi}\, t^{b-1}\sin(t\sin\phi)\, dt = [\Gamma(b)]\sin(b\phi), \qquad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$

81. $\int_0^\infty e^{-t\cos\phi}\, t^{b-1}[\cos(t\sin\phi)]\, dt = [\Gamma(b)]\cos(b\phi), \qquad \left(b > 0, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$

82. $\int_0^\infty t^{b-1}\cos t\, dt = [\Gamma(b)]\cos\left(\frac{b\pi}{2}\right), \qquad (0 < b < 1)$

83. $\int_0^\infty t^{b-1}(\sin t)\, dt = [\Gamma(b)]\sin\left(\frac{b\pi}{2}\right), \qquad (0 < b < 1)$

84. $\int_0^1 (\log x)^n\, dx = (-1)^n \cdot n!$

85. $\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$

86. $\int_0^1 \left(\log\frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$

87. $\int_0^1 \left(\log\frac{1}{x}\right)^n dx = n!$

88. $\int_0^1 x\log(1-x)\, dx = -\frac{3}{4}$

89. $\int_0^1 x\log(1+x)\, dx = \frac{1}{4}$

90. $\int_0^1 x^m(\log x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \qquad m > -1, n = 0, 1, 2, \ldots$

If $n \neq 0, 1, 2, \ldots$ replace $n!$ by $\Gamma(n+1)$.

DEFINITE INTEGRALS (Continued)

691. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$

692. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$

693. $\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{12}$

694. $\int_0^1 \frac{\log(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

695. $\int_0^1 (\log x)[\log(1+x)]\,dx = 2 - 2\log 2 - \frac{\pi^2}{12}$

696. $\int_0^1 (\log x)[\log(1-x)]\,dx = 2 - \frac{\pi^2}{6}$

697. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}$

698. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}$

699. $\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2$

700. $\int_0^1 x^m\left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, if $m+1>0, n+1>0$

701. $\int_0^1 \frac{(x^p - x^q)\,dx}{\log x} = \log\left(\frac{p+1}{q+1}\right)$, $(p+1>0, q+1>0)$

702. $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}$, (same as integral 686)

703. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}$

704. $\int_0^{\pi/2} (\log \sin x)\,dx = \int_0^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2$

705. $\int_0^{\pi/2} (\log \sec x)\,dx = \int_0^{\pi/2} \log \csc x\,dx = \frac{\pi}{2}\log 2$

706. $\int_0^\pi x(\log \sin x)\,dx = -\frac{\pi^2}{2}\log 2$

707. $\int_0^{\pi/2} (\sin x)(\log \sin x)\,dx = \log 2 - 1$

DEFINITE INTEGRALS (Continued)

708. $$\int_0^{\pi/2} (\log \tan x)\, dx = 0$$

709. $$\int_0^{\pi} \log (a \pm b \cos x)\, dx = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2}\right), \qquad (a \geq b)$$

710. $$\int_0^{\pi} \log (a^2 - 2ab \cos x + b^2)\, dx = \begin{cases} 2\pi \log a, & a \geq b > 0 \\ 2\pi \log b, & b \geq a > 0 \end{cases}$$

711. $$\int_0^{\infty} \frac{\sin ax}{\sinh bx}\, dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b}$$

712. $$\int_0^{\infty} \frac{\cos ax}{\cosh bx}\, dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}$$

713. $$\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a}$$

714. $$\int_0^{\infty} \frac{x\, dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

715. $$\int_0^{\infty} e^{-ax}(\cosh bx)\, dx = \frac{a}{a^2 - b^2}, \qquad (0 \leq |b| < a)$$

716. $$\int_0^{\infty} e^{-ax}(\sinh bx)\, dx = \frac{b}{a^2 - b^2}, \qquad (0 \leq |b| < a)$$

717. $$\int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1}\, dx = \frac{\pi}{2b} \csc \frac{a\pi}{b} - \frac{1}{2a}$$

718. $$\int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1}\, dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$$

719. $$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots\right], \text{ if } k^2 < 1$$

720. $$\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\, dx = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right], \text{ if } k^2 < 1$$

721. $$\int_0^{\infty} e^{-x} \log x\, dx = -\gamma = -0.5772157\ldots$$

722. $$\int_0^{\infty} e^{-x^2} \log x\, dx = -\frac{\sqrt{\pi}}{4}(\gamma + 2 \log 2)$$

DEFINITE INTEGRALS (Continued)

723. $\int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x}\, dx = \gamma = 0.5772157\ldots$ [Euler's Constant]

724. $\int_0^\infty \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right) dx = \gamma = 0.5772157\ldots$

For n even:

725. $\int \cos^n x\, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \frac{\sin (n-2k)\, x}{(n-2k)} + \frac{1}{2^n}\binom{n}{\frac{n}{2}} x$

726. $\int \sin^n x\, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} \frac{\sin [(n-2k)(\frac{\pi}{2}-x)]}{2k-n} + \frac{1}{2^n}\binom{n}{\frac{n}{2}} x$

For n odd:

727. $\int \cos^n x\, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{\sin (n-2k)\, x}{(n-2k)}$

728. $\int \sin^n x\, dx = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{\sin [n-2k)(\frac{\pi}{2}-x)]}{2k-n}$

Volume by double Integration

$V = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\, dx\, dy$ $x = h(y)$ + $x = g(y)$, $y = a$ and $y = b$

Vol. of 1st octant of a sphere whose equa = $x^2 + y^2 + z^2 =$
Can be done 2 ways

(1) $V = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx\, dy$

or by double summation:

(2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} z\, dy\, dy$ $z = \sqrt{a^2-y^2-x^2}$

Volume by Triple Integration:

of a sphere:

$\int^a \int^{\sqrt{a^2-y^2}} \int^{\sqrt{a^2-y^2-x^2}} dz\, dx\, dy$

SERIES EXPANSION

The expression in parentheses following certain of the series indicates the region of con-
rgence. If not otherwise indicated it is to be understood that the series converges for all
ite values of x.

BINOMIAL

$$+ y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots \quad (y^2 < x^2)$$

$$\pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \text{etc.} \quad (x^2 < 1)$$

$$\pm x)^{-n} = 1 \mp nx + \frac{n(n+1)x^2}{2!} \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots \text{etc.} \quad (x^2 < 1)$$

$$\pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots \quad (x^2 < 1)$$

$$\pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \cdots \quad (x^2 < 1)$$

REVERSION OF SERIES

Let a series be represented by

$$y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \quad (a_1 \neq 0)$$

find the coefficients of the series

$$x = A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \cdots$$

$$= \frac{1}{a_1} \qquad A_2 = -\frac{a_2}{a_1^3} \qquad A_3 = \frac{1}{a_1^5}(2a_2^2 - a_1a_3)$$

$$= \frac{1}{a_1^7}(5a_1a_2a_3 - a_1^2a_4 - 5a_2^3)$$

$$= \frac{1}{a_1^9}(6a_1^2a_2a_4 + 3a_1^2a_3^2 + 14a_2^4 - a_1^3a_5 - 21a_1a_2^2a_3)$$

$$= \frac{1}{a_1^{11}}(7a_1^3a_2a_5 + 7a_1^3a_3a_4 + 84a_1a_2^3a_3 - a_1^4a_6 - 28a_1^2a_2^2a_4 - 28a_1^2a_2a_3^2 - 42a_2^5)$$

$$= \frac{1}{a_1^{13}}(8a_1^4a_2a_6 + 8a_1^4a_3a_5 + 4a_1^4a_4^2 + 120a_1^2a_2^3a_4 + 180a_1^2a_2^2a_3^2 + 132a_2^6 - a_1^5a_7 - 36a_1^3a_2^2a_5 - 72a_1^3a_2a_3a_4 - 12a_1^3a_3^3 - 330a_1a_2^4a_3)$$

TAYLOR

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots \text{ (Taylor's Series)}$$

Complex Roots $= e^{ax}[C_1 \sin ix + C_2 \cos ix]$

Real roots $= C_1e^{m_1x} + C_2e^{m_2x}$

Repeated roots $= C_1e^{mx} + C_2xe^{mx}$

(Increment form)

2. $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$$= f(h) + xf'(h) + \frac{x^3}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots$$

3. If $f(x)$ is a function possessing derivatives of all orders throughout the interva $a \leqq x \leqq b$, then there is a value X, with $a < X < b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(X)$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h), \quad b = a+h, \; 0 < \theta <$$

or

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + (x-a)^{n-1} \frac{f^{(n-1)}(a)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{f^{(n)}[a + \theta\cdot(x-a)]}{n!} (x-a)^n, \quad 0 < \theta < 1.$$

The above forms are known as Taylor's series with the remainder term.

4. *Taylor's series for a function of two variables*

If $\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(x,y) = h \frac{\partial f(x,y)}{\partial x} + k \frac{\partial f(x,y)}{\partial y}$;

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 f(x,y) = h^2 \frac{\partial^2 f(x,y)}{\partial x^2} + 2hk \frac{\partial^2 f(x,y)}{\partial x \, \partial y} + k^2 \frac{\partial^2 f(x,y)}{\partial y^2}$$

etc., and if $\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}}$ with the bar and subscripts means that after differentiation we are to replace x by a and y by b,

$$f(a+h, b+k) = f(a,b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right) f(x,y)\Big|_{\substack{x=a\\y=b}} + \cdots + \frac{1}{n!}\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f(x,y)\Big|_{\substack{x=a\\y=b}} + \cdots$$

MACLAURIN

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + x^{n-1} \frac{f^{(n-1)}(0)}{(n-1)!} + R_n,$$

where

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}, \quad 0 < \theta < 1.$$

EXPONENTIAL

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$= 1 + x \log_e a + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \cdots$$

$$= e^a \left[1 + (x - a) + \frac{(x - a)^2}{2!} + \frac{(x - a)^3}{3!} + \cdots\right]$$

LOGARITHMIC

$$\log_e x = \frac{x - 1}{x} + \frac{1}{2}\left(\frac{x - 1}{x}\right)^2 + \frac{1}{3}\left(\frac{x - 1}{x}\right)^3 + \cdots \qquad (x > \tfrac{1}{2})$$

$$\log_e x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \cdots \qquad (2 \geq x > 0)$$

$$\log_e x = 2\left[\frac{x - 1}{x + 1} + \frac{1}{3}\left(\frac{x - 1}{x + 1}\right)^3 + \frac{1}{5}\left(\frac{x - 1}{x + 1}\right)^5 + \cdots\right] \qquad (x > 0)$$

$$\log_e(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \qquad (-1 < x \leqslant 1)$$

$$\log_e(n + 1) - \log_e(n - 1) = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right]$$

$$\log_e(a + x) = \log_e a + 2\left[\frac{x}{2a + x} + \frac{1}{3}\left(\frac{x}{2a + x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a + x}\right)^5 + \cdots\right]$$

$$(a > 0,\ -a < x < +\infty)$$

$$\log_e \frac{1 + x}{1 - x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n - 1} + \cdots\right], \qquad -1 < x < 1$$

$$\log_e x = \log_e a + \frac{(x - a)}{a} - \frac{(x - a)^2}{2a^2} + \frac{(x - a)^3}{3a^3} - + \cdots, \qquad 0 < x \leqq 2a$$

TRIGONOMETRIC

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{(all real values of } x\text{)}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots + \frac{(-1)^{n-1}\, 2^{2n}\,(2^{2n} - 1)\, B_{2n}}{(2n)!} x^{2n-1} + \cdots,$$

$\left[x^2 < \frac{\pi^2}{4}\right.$, and B_n represents the n'th Bernoulli number.$\left.\right]$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \cdots - \frac{(-1)^{n+1} 2^{2n}}{(2n)!} B_{2n} x^{2n-1} - \cdots,$$

[$x^2 < \pi^2$, and B_n represents the n'th Bernoulli number.]

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots + \frac{(-1)^n}{(2n)!}E_{2n}x^{2n} + \cdots,$$

$\left[x^2 < \frac{\pi^2}{4}\right.$, and E_n represents the n'th Euler number.

$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15{,}120}x^5 + \frac{127}{604{,}800}x^7 + \cdots$$

$$+ \frac{(-1)^{n+1}2(2^{2n-1}-1)}{(2n)!}B_{2n}x^{2n-1} + \cdots,$$

[$x^2 < \pi^2$, and B_n represents n'th Bernoulli number.]

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \qquad (x^2 < \infty)$$

$$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\cdots \qquad (x^2 < \infty)$$

$$\sin^{-1} x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots \qquad \left(x^2 < 1, -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}\right)$$

$$\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots\right) \qquad (x^2 < 1, 0 < \cos^{-1} x < \pi)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (x^2 < 1)$$

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \cdots \qquad (x > 1)$$

$$\tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \cdots \qquad (x < -1)$$

$$\cot^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots \qquad (x^2 < 1)$$

$$\log_e \sin x = \log_e x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots \qquad (x^2 < \pi^2)$$

$$\log_e \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$\log_e \tan x = \log_e x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \cdots$$

$$e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right)$$

$$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

$$\sin x = \sin a + (x - a)\cos a - \frac{(x-a)^2}{2!}\sin a$$

$$- \frac{(x-a)^3}{3!}\cos a + \frac{(x-a)^4}{4!}\sin a + \cdots$$

HYPERBOLIC AND INVERSE HYPERBOLIC

See page 263 for section on Hyperbolic Functions

See page 268 for section on Hyperbolic Series

FOURIER

(Also see Index for Cosine and Sine Transforms)

1. If $f(x)$ is a bounded periodic function of period $2L$ (i.e. $f(x + 2L) = f(x)$), and satisfies the *Dirichlet conditions*:

a) In any period $f(x)$ is continuous, except possibly for a finite number of jump discontinuities

b) In any period $f(x)$ has only a finite number of maxima and minima.

Then $f(x)$ may be represented by the *Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

where a_n and b_n are as determined below. This series will converge to $f(x)$ at every point where $f(x)$ is continuous, and to

$$\frac{f(x^+) + f(x^-)}{2}$$

(i.e. the average of the left-hand and right-hand limits) at every point where $f(x)$ has a jump discontinuity.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, 3, \ldots;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

We may also write

$$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\frac{n\pi x}{L}\,dx \text{ and } b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\frac{n\pi x}{L}\,dx,$$

where α is any real number. Thus if $\alpha = 0$,

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, 3, \ldots;$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

2. If in addition to the above restrictions, $f(x)$ is even (i.e. $f(-x) = f(x)$), the Fourier series reduces to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$$